Provers & Solvers

Lecture 3: λ-Superposition

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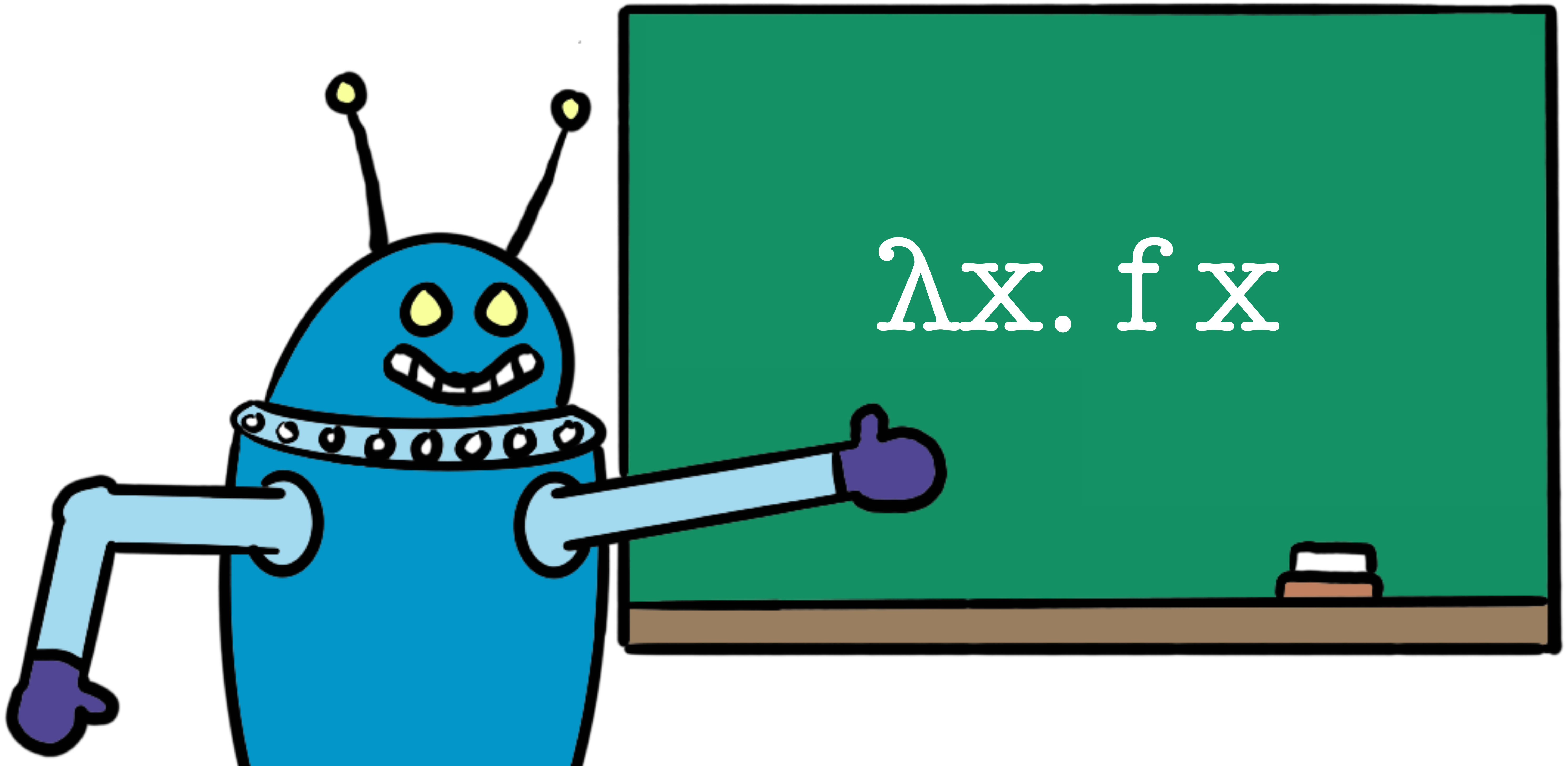
Partly based on slides by
Alexander Bentkamp

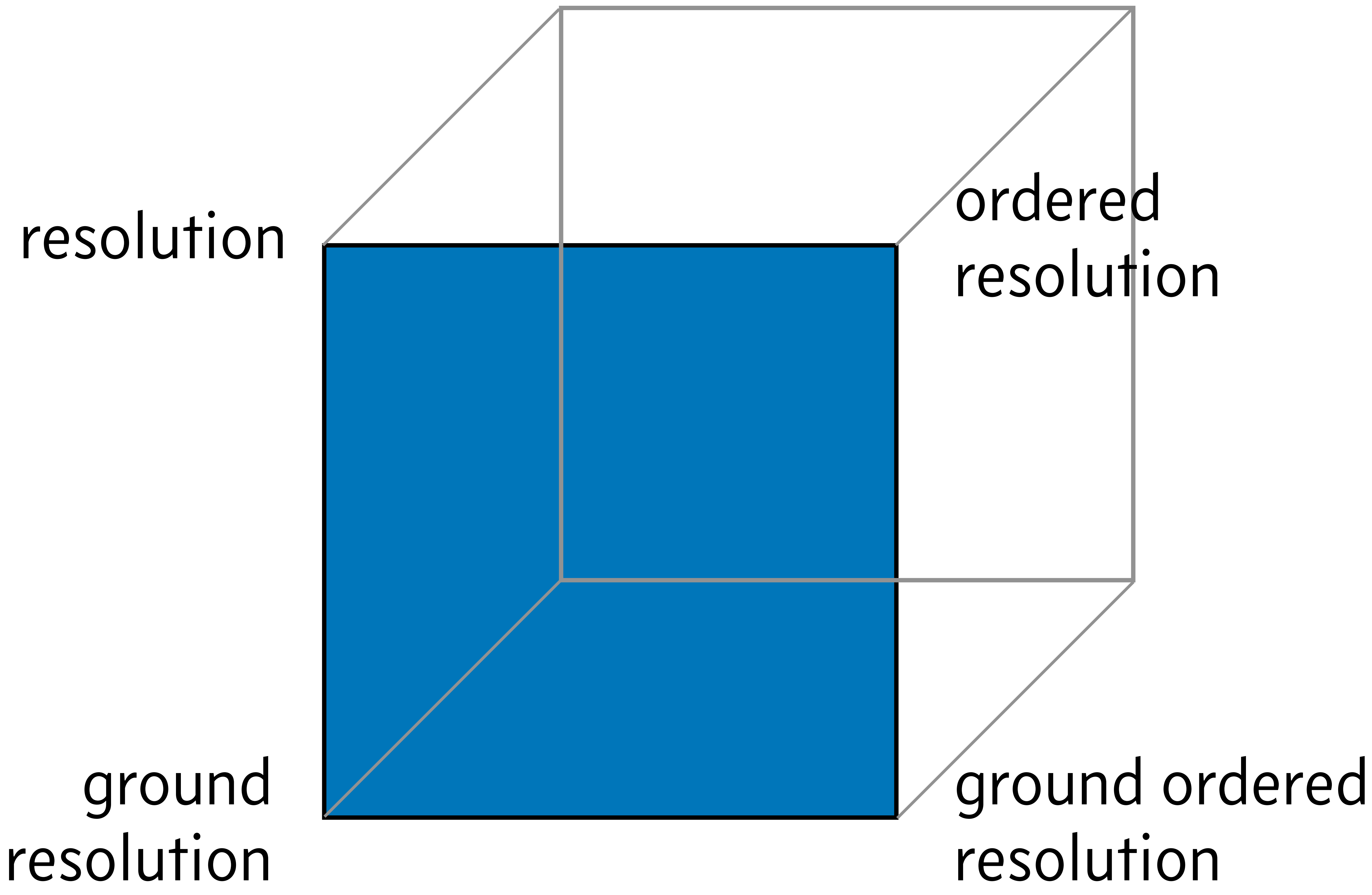


Outline of These Lectures

1. Resolution
2. Superposition
3. **λ-Superposition**
4. CDCL and CDCL(T)
5. AVATAR

Disclaimer

Relations between the Calculi



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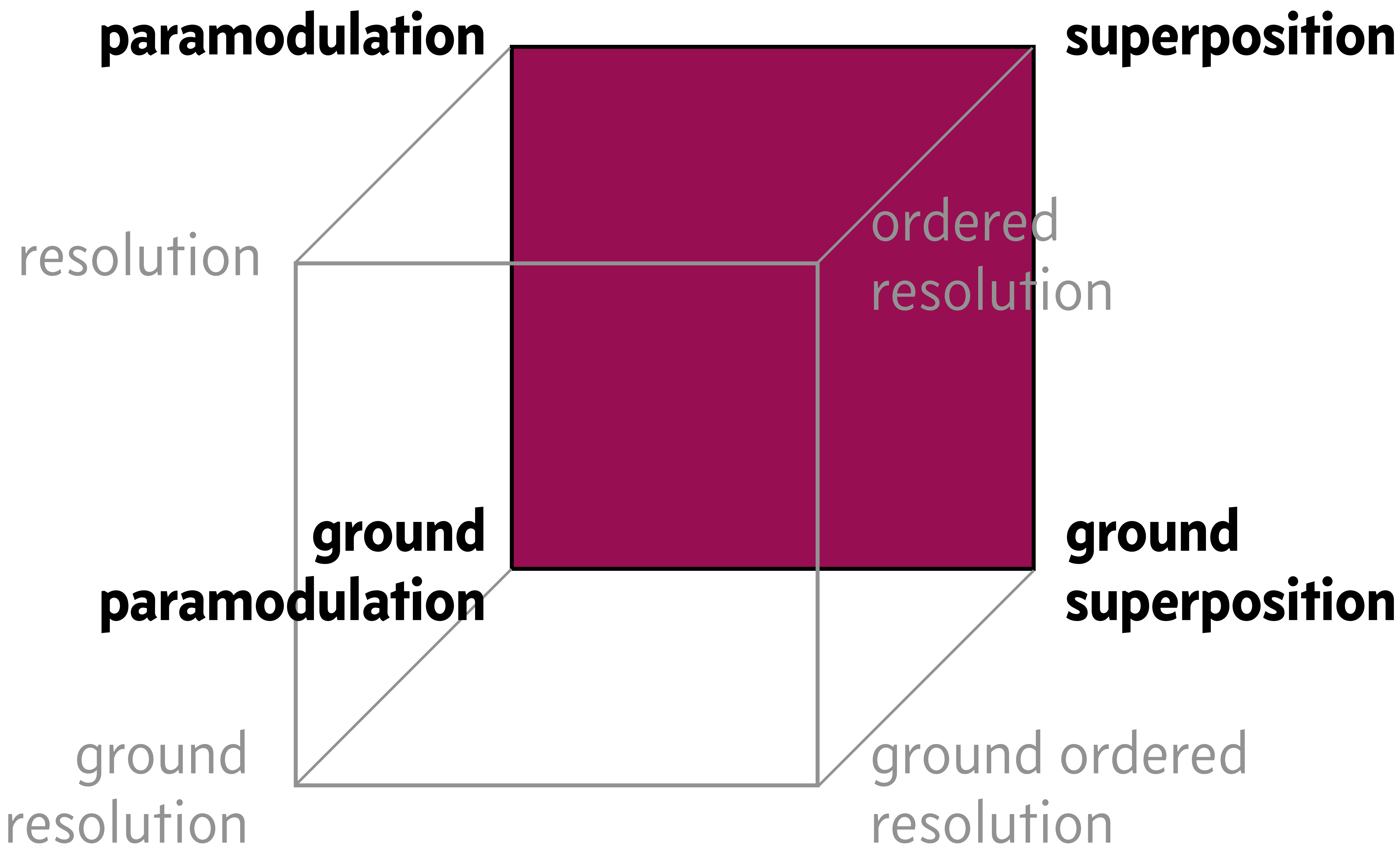

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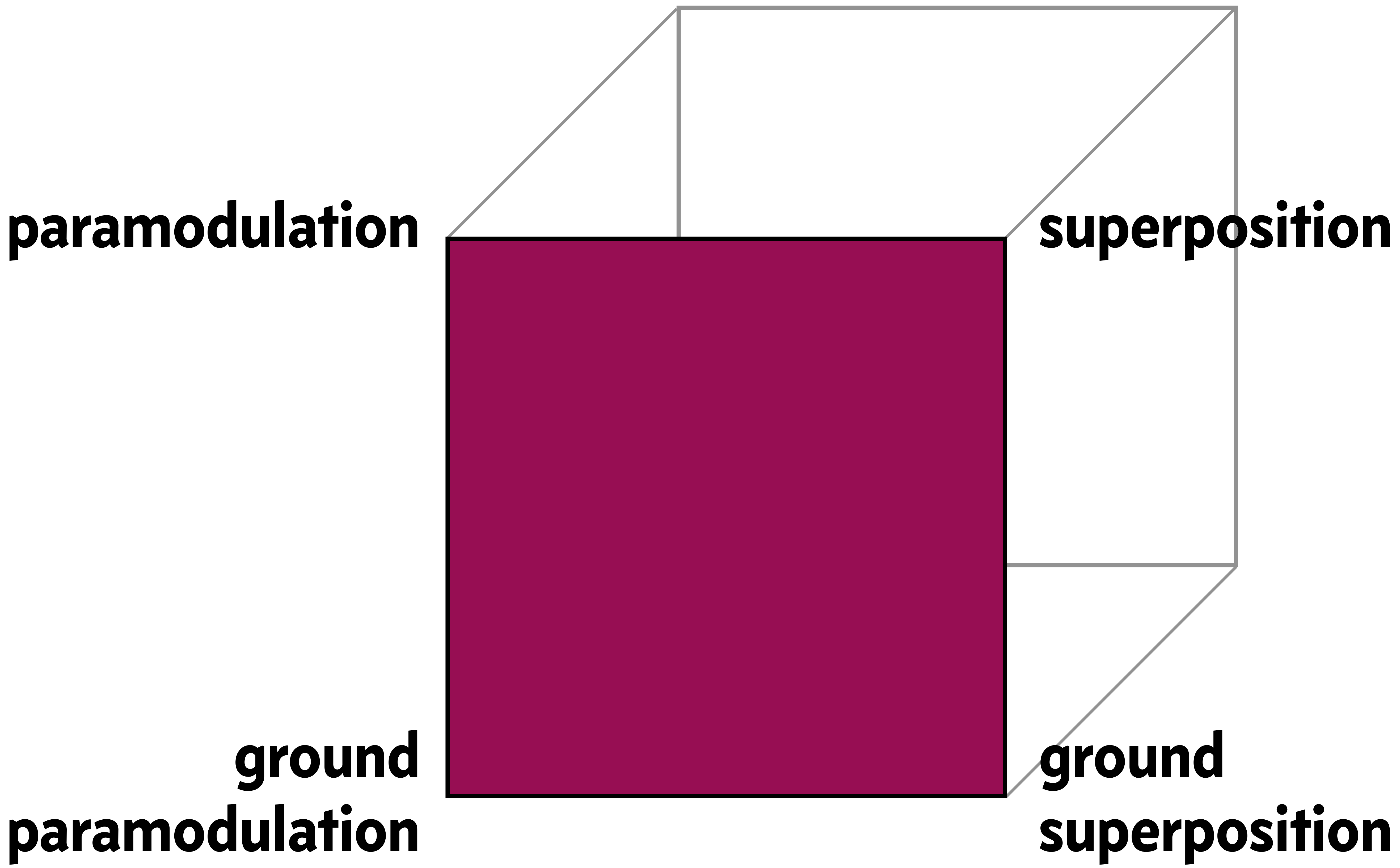

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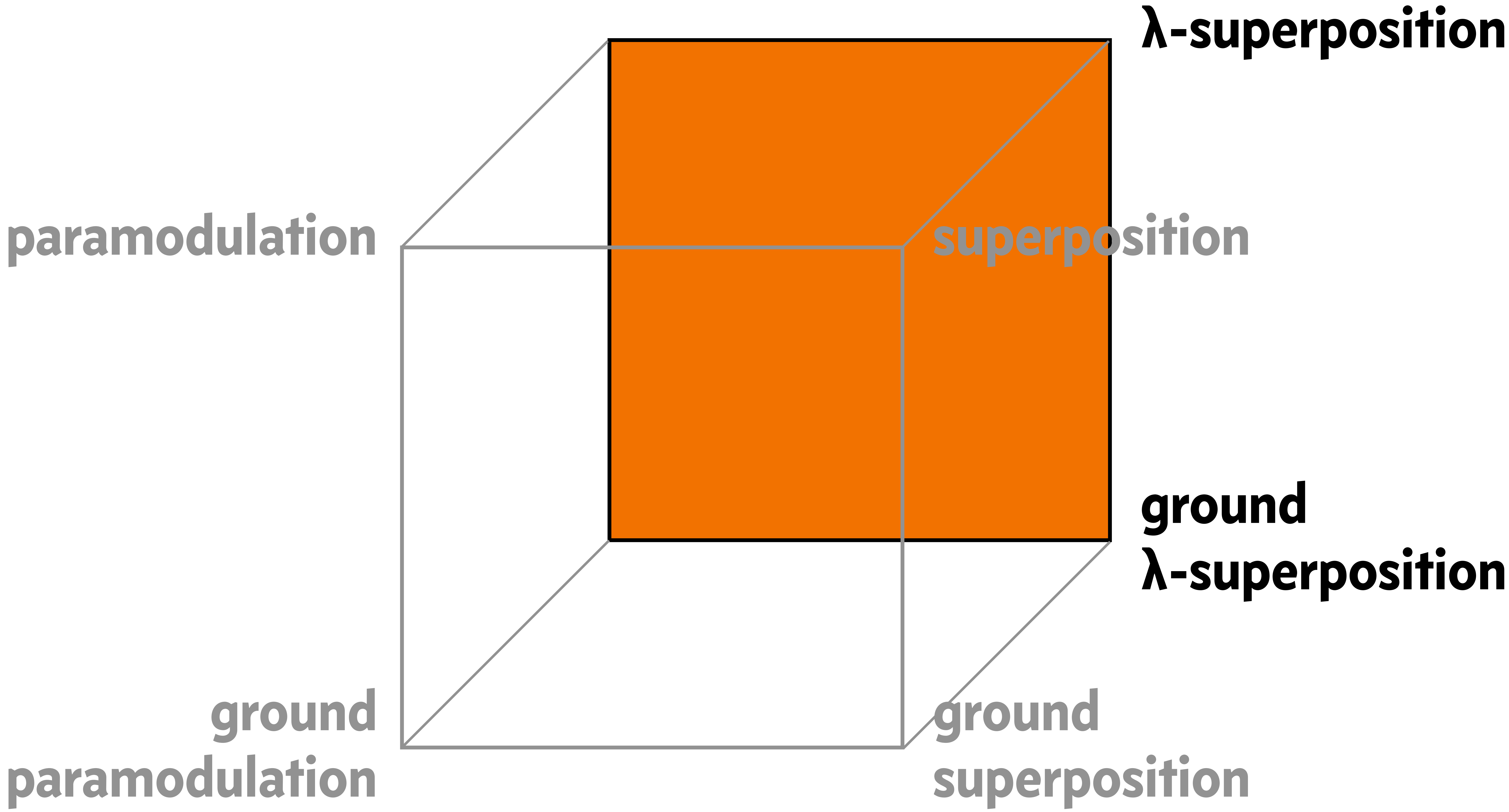

A Higher-Order Proof Goal

Proof assistant

show $\left(\sum_{i=1}^{n} i^2 + 2i + 1\right) = \left(\sum_{i=1}^{n} i^2\right) + \left(\sum_{i=1}^{n} 2i\right) + \left(\sum_{i=1}^{n} 1\right)$

Find Proof

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Find Proof

Lost in Translation

$$\left(\sum_{i=1}^{n} i^2 + 2i + 1\right) = \left(\sum_{i=1}^{n} i^2\right) + \left(\sum_{i=1}^{n} 2i\right) + \left(\sum_{i=1}^{n} 1\right)$$

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sum(1, n, C(B(plus, S(B(plus, C(power, 2)), app(times, 2))), 1))
= app(app(plus, app(app(plus, sum(1, n, C(power, 2))),
sum(1, n, app(times, 2)))), sum(1, n, K(1)))

Design Goal for λ-Superposition

A sound, complete, **graceful generalization**
of first-order superposition

Syntax of Higher-Order Logic

Given a **signature**, consisting of

- atomic types (e.g., bool, nat)
- symbols (e.g., 1, ·, gcd) declared with their types

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The **terms** are defined by these rules:

- A variable x declared with type τ is a term of type τ
- A symbol f declared with type τ is a term of type τ
- If t has type τ, an abstraction $\lambda x : \sigma.\ t$ is a term of type σ → τ
- If t has type σ → τ and t' has type σ, an application $t\ t'$ is a term of type τ

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Formulas are terms of type bool

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Examples: Higher-Order Terms

g a x

f (g a x) b y

p f (f a) (f a b)

$\lambda x.\ \lambda y.$ g y x

1 + 2

αβη-Equivalence

Terms are considered equal up to the following three rules:

(α) $(\lambda x.\ t(x)) = (\lambda y.\ t(y))$

(β) $(\lambda x.\ t(x))\ u = t(u)$ if x does not occur free in u

(η) $(\lambda x. \mathrm{t}\ \mathrm{x}) = t$ if x does not occur free in t

Examples: αβη-Equivalence

$$g\ (\lambda x.\ f\ x\ x) = g\ (\lambda z.\ f\ z\ z)$$

$$(\lambda x.\ f\ x\ x)\ a = f\ a\ a$$

$$g\ (f\ a\ a) = g\ (\lambda y.\ f\ a\ a\ y)$$

Syntax of Clausal Higher-Order Logic

The **atoms** are defined by this rule:

- If t_1 and t_2 are terms, then $t_1 = t_2$ (viewed as an unordered pair) is an atom

The **literals** are defined by this rule:

- If A is an atom, then A and $\neg A$ are literals

The **clauses** are defined by this rule:

- If $L_1, \ldots, L_n$ are literals, then $L_1 \vee \cdots \vee L_n$ (viewed as a multiset) is a clause

We write $\bot$ if $n = 0$

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- If t_1 and t_2 are terms, then $t_1 = t_2$ (viewed as an unordered pair) is an atom

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We write $\bot$ if $n = 0$

- $\top$, $\land$, $\Rightarrow$, $\forall$, $\exists$ can appear in terms
- Variables are understood as "for all"

Example: Summations

Goal:

$$\left(\sum_{i=1}^{n} i^2 + 2i + 1\right) = \left(\sum_{i=1}^{n} i^2\right) + \left(\sum_{i=1}^{n} 2i\right) + \left(\sum_{i=1}^{n} 1\right)$$

Distributivity lemma:

$$\forall f, g\,.\ \sum_{i=1}^{n} \left(f\ i + g\ i\right) = \sum_{i=1}^{n} f\ i + \sum_{i=1}^{n} g\ i$$

Proof idea:

$$\left(\sum_{i=1}^{n} i^2 + 2i + 1\right) = \sum_{i=1}^{n} \left(i^2 + 2i\right) + \sum_{i=1}^{n} 1 = \left(\sum_{i=1}^{n} i^2\right) + \left(\sum_{i=1}^{n} 2i\right) + \left(\sum_{i=1}^{n} 1\right)$$

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sum 1 n (λ*i*. *i*² + 2*i* + 1) = sum 1 n (λ*i*. *i*²) + sum 1 n (λ*i*. 2*i*) + sum 1 n (λ*i*. 1)

Distributivity lemma:

$$\forall f, g\,.\ \sum_{i=1}^{n} (f\,i + g\,i) = \sum_{i=1}^{n} f\,i + \sum_{i=1}^{n} g\,i$$

sum 1 n (λ*i*. *f i* + *g i*) = sum 1 n (λ*i*. *f i*) + sum 1 n (λ*i*. *g i*)

Proof idea:

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Example: Summations

$$\text{sum } 1 \text{ n } (\lambda i.\, f\, i + g\, i)$$
$$= \text{sum } 1 \text{ n } (\lambda i.\, f\, i) + \text{sum } 1 \text{ n } (\lambda i.\, g\, i)$$

$$\text{sum } 1 \text{ n } (\lambda i.\, i^2 + 2i + 1)$$
$$\neq \text{sum } 1 \text{ n } (\lambda i.\, i^2) + \text{sum } 1 \text{ n } (\lambda i.\, 2i) + \text{sum } 1 \text{ n } (\lambda i.\, 1)$$

Example: Summations

sum 1 n ($\lambda i.\, f\, i + g\, i$)
= sum 1 n ($\lambda i.\, f\, i$) + sum 1 n ($\lambda i.\, g\, i$)

sum 1 n ($\lambda i.\, i^2 + 2i + 1$)
≠ sum 1 n ($\lambda i.\, i^2$) + sum 1 n ($\lambda i.\, 2i$) + sum 1 n ($\lambda i.\, 1$)

sum 1 n ($\lambda i.\, i^2 + 2i + 1$)
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$\bot$

Design Challenges for λ-Superposition

1. The term order
2. Unification
3. Booleans

Challenge 1: The Term Order

In first-order superposition:

- A subterm must be smaller than the whole term:
  e.g. $c < g(c) < f(g(c))$
- Putting two terms in the same context should preserve the orientation:
  e.g. if $b > a$ then $f(b, c) > f(a, c)$

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Issue in higher-order:

$$(\lambda y.\ f\ (\lambda z.\ z)) \quad > \quad (\lambda z.\ z)$$

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$$[\,]\ (g\ (f\ (\lambda z.\ z))) \qquad [\,]\ (g\ (f\ (\lambda z.\ z)))$$

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$\mathrm{f}\ (\lambda z.\ z)$

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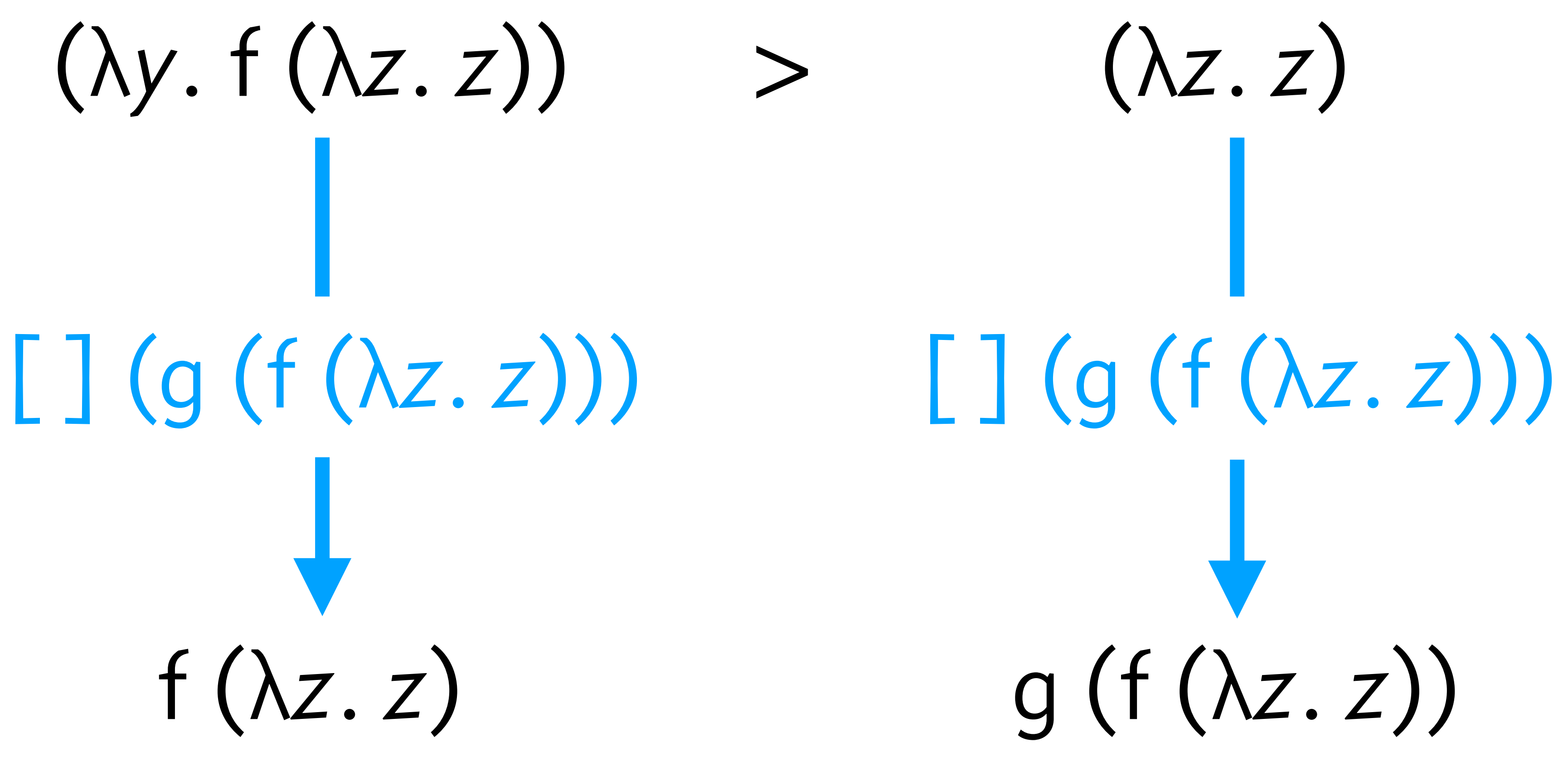

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$$(\lambda y.\, \mathrm{f}\,(\lambda z.\, z)) \quad > \quad (\lambda z.\, z)$$

$$[\,]\,(\mathrm{g}\,(\mathrm{f}\,(\lambda z.\, z))) \qquad [\,]\,(\mathrm{g}\,(\mathrm{f}\,(\lambda z.\, z)))$$

$$\mathrm{f}\,(\lambda z.\, z) \quad < \quad \mathrm{g}\,(\mathrm{f}\,(\lambda z.\, z))$$

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$$\downarrow\ [\,]\ (\mathrm{g}\ (\mathrm{f}\ (\lambda z.\ z))) \qquad\qquad \downarrow\ [\,]\ (\mathrm{g}\ (\mathrm{f}\ (\lambda z.\ z)))$$

$$\mathrm{f}\ (\lambda z.\ z) \quad \not> \quad \mathrm{g}\ (\mathrm{f}\ (\lambda z.\ z))$$

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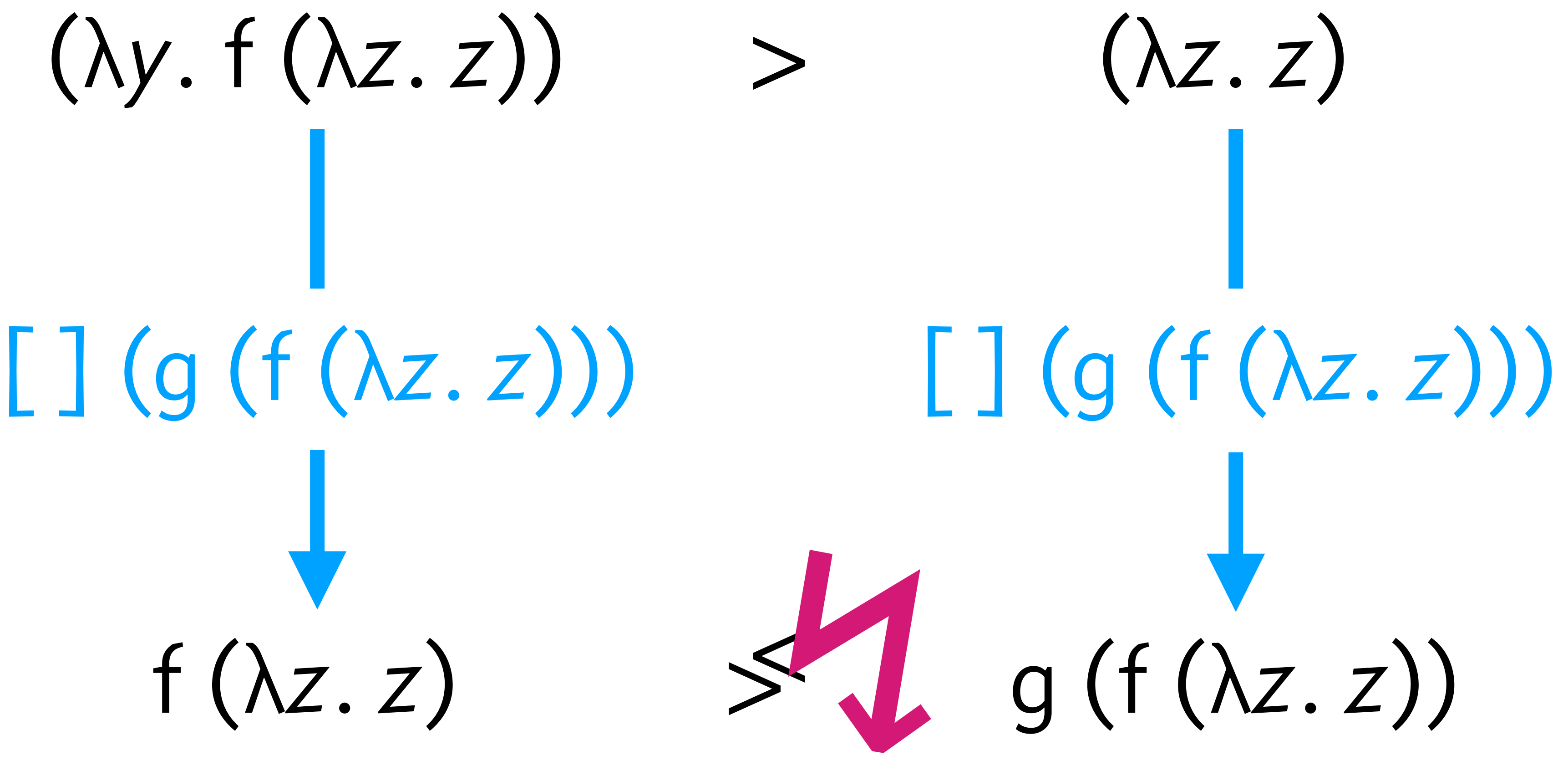

Challenge 1: The Term Order

Solution:

- Weaken the second requirement: only **good contexts** must preserve the orientation
- The main superposition rule acts only on **good contexts**
- Compensate with an **extra calculus rule**

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- The main superposition rule acts only on **good contexts**
- Compensate with an **extra calculus rule**

Extra rule:

If the clause

$$C \vee t = t'$$

is contained in $\mathbf{F}$
and t, t' are functions,
then add the clause

$$C \vee t\ x = t'\ x$$

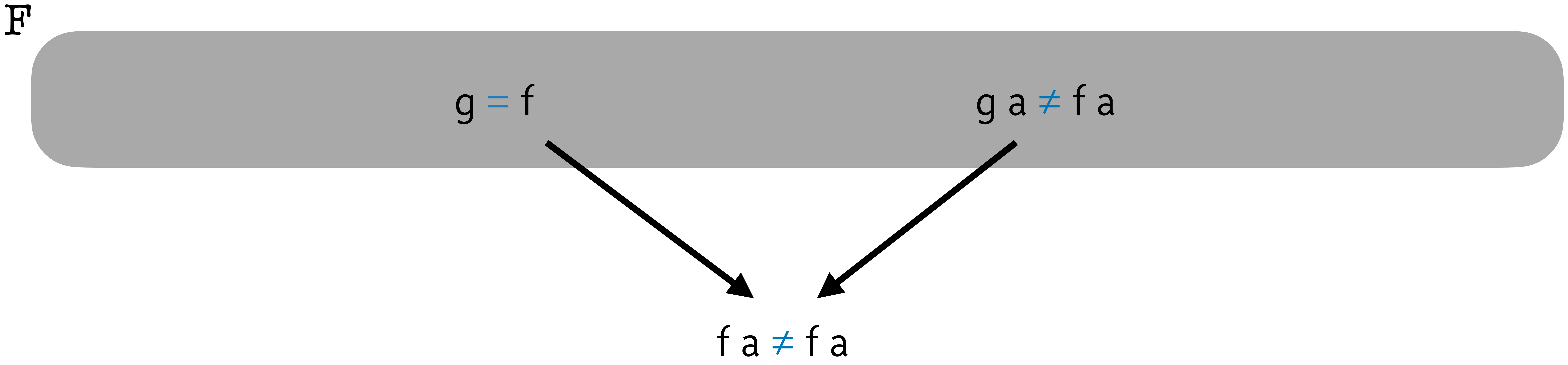
to $\mathbf{F}$

Example: The Term Order

F

$g = f$ $\quad$ $g\ a \neq f\ a$

Example: The Term Order



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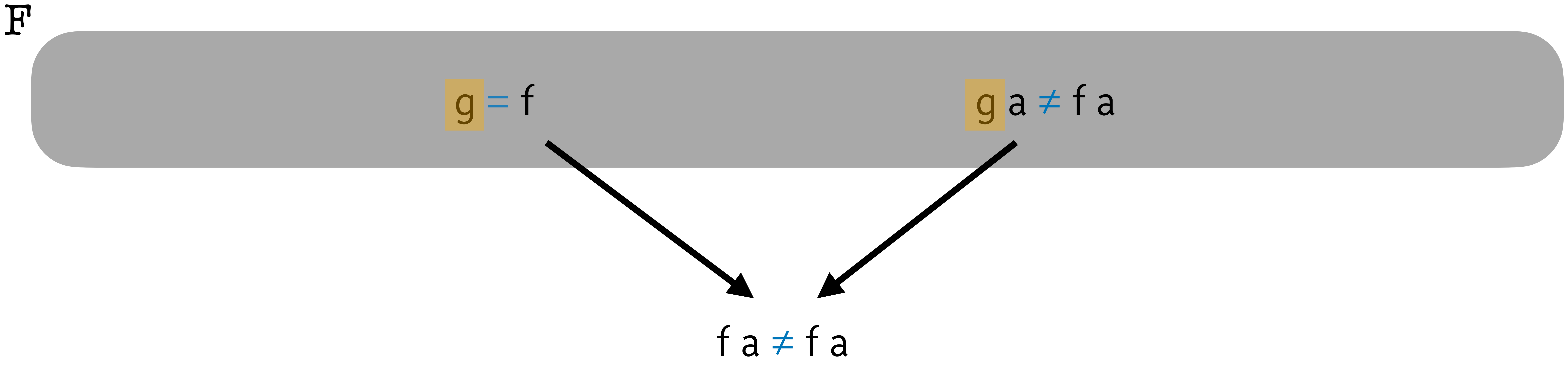

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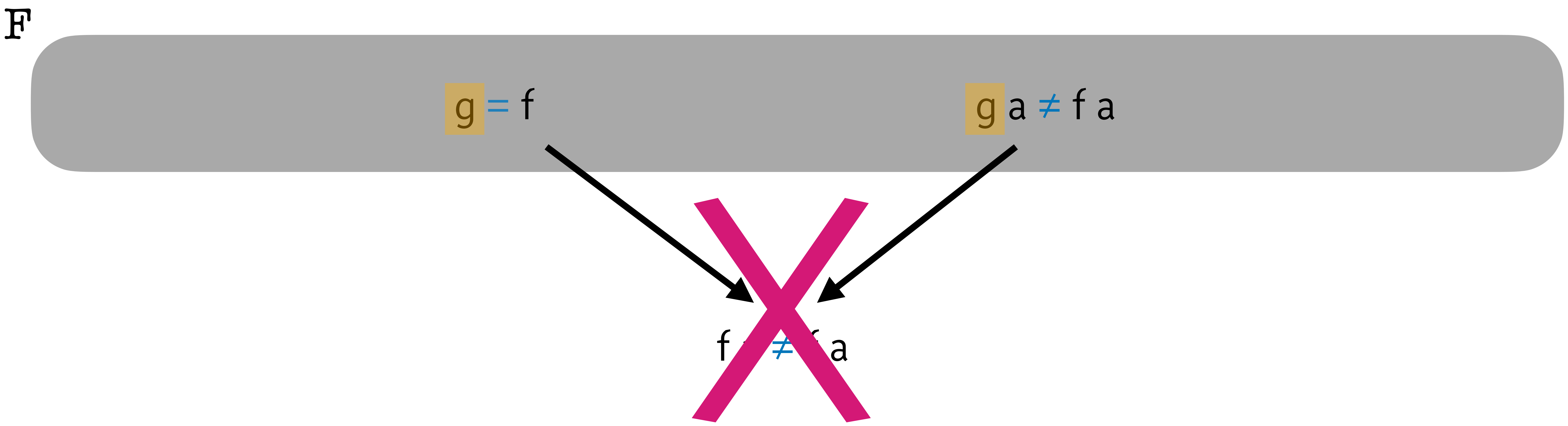


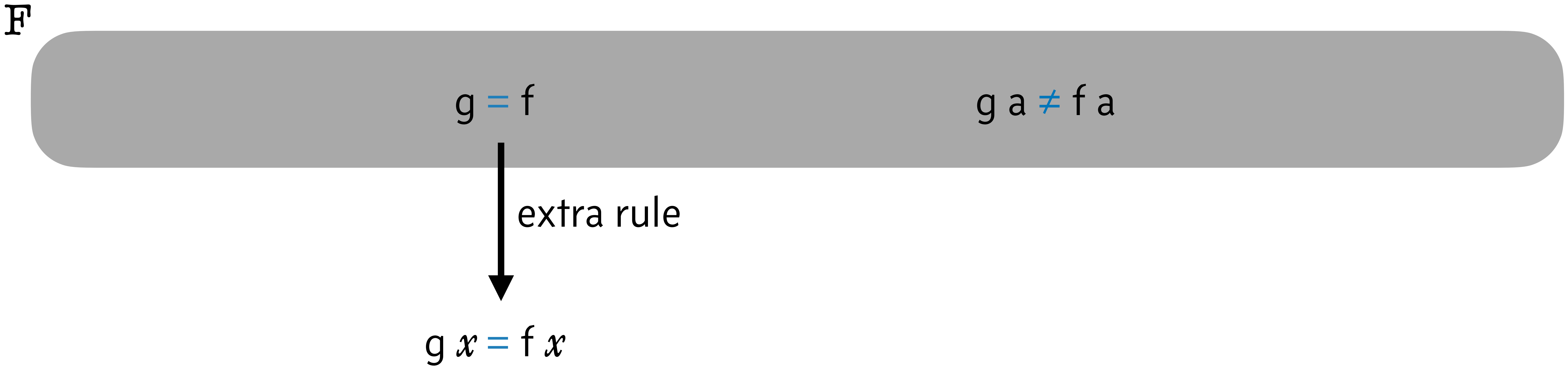
disallowed because [ ] a is not a good context

Example: The Term Order

F

$g = f \qquad g\ a \neq f\ a$

Example: The Term Order



Example: The Term Order

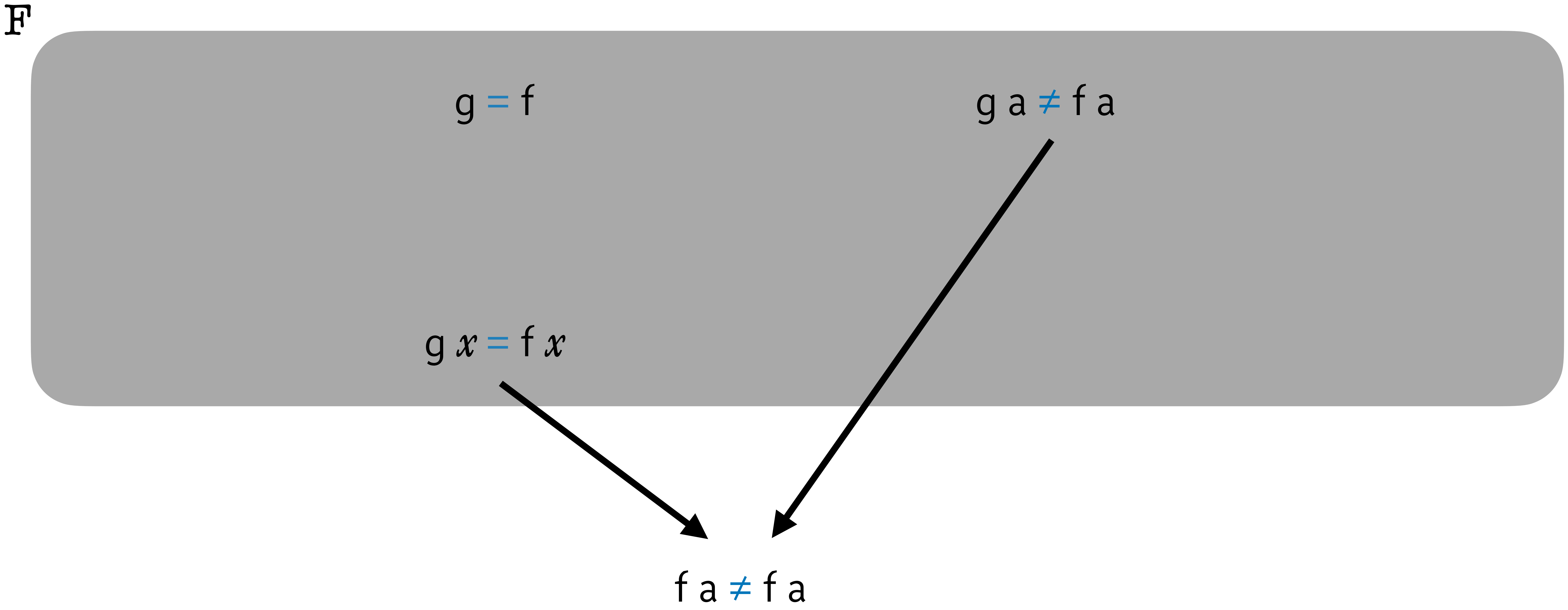
$\mathtt{F}$

$\mathsf{g} = \mathsf{f}$

$\mathsf{g}\ \mathsf{a} \neq \mathsf{f}\ \mathsf{a}$

$\mathsf{g}\ x = \mathsf{f}\ x$

Example: The Term Order



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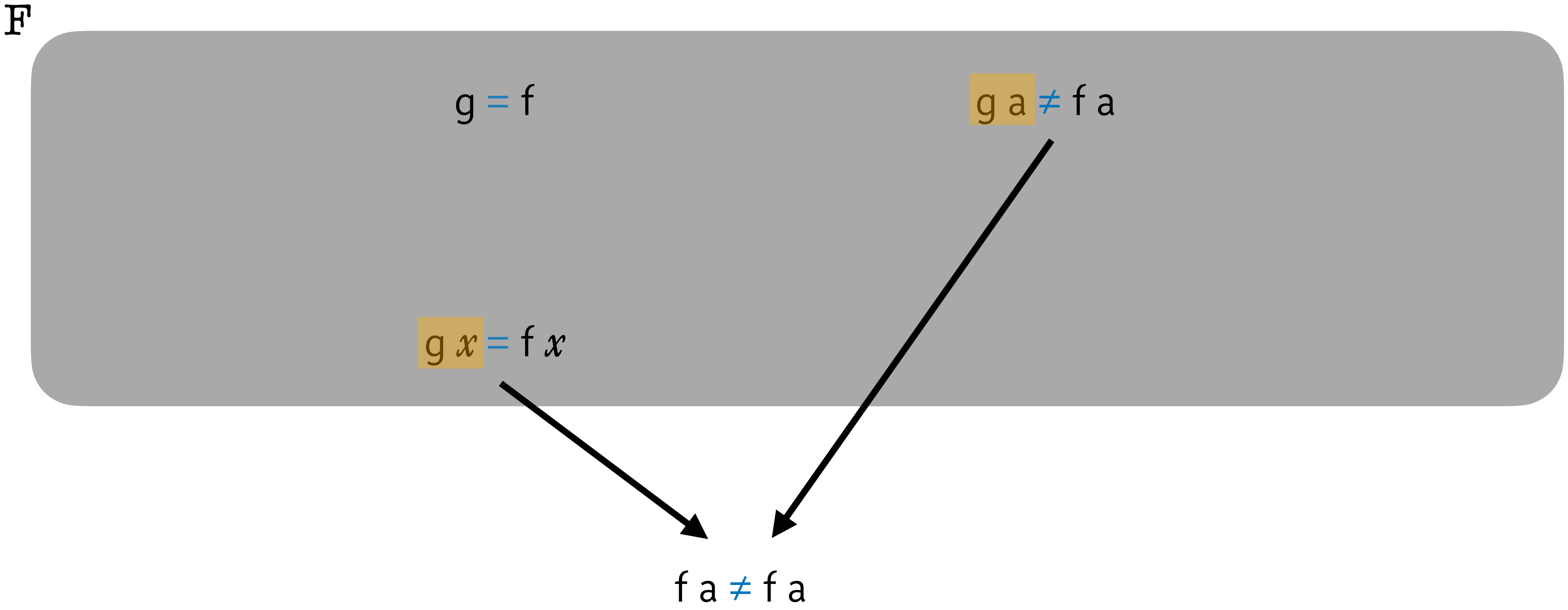

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Challenge 1bis: The Redundancy Criterion

In first-order logic:

- A ground clause C is redundant w.r.t. ground $\mathtt{F}$ if $C_1, \ldots, C_n \in \mathtt{F}$ are all smaller than C and they together entail C

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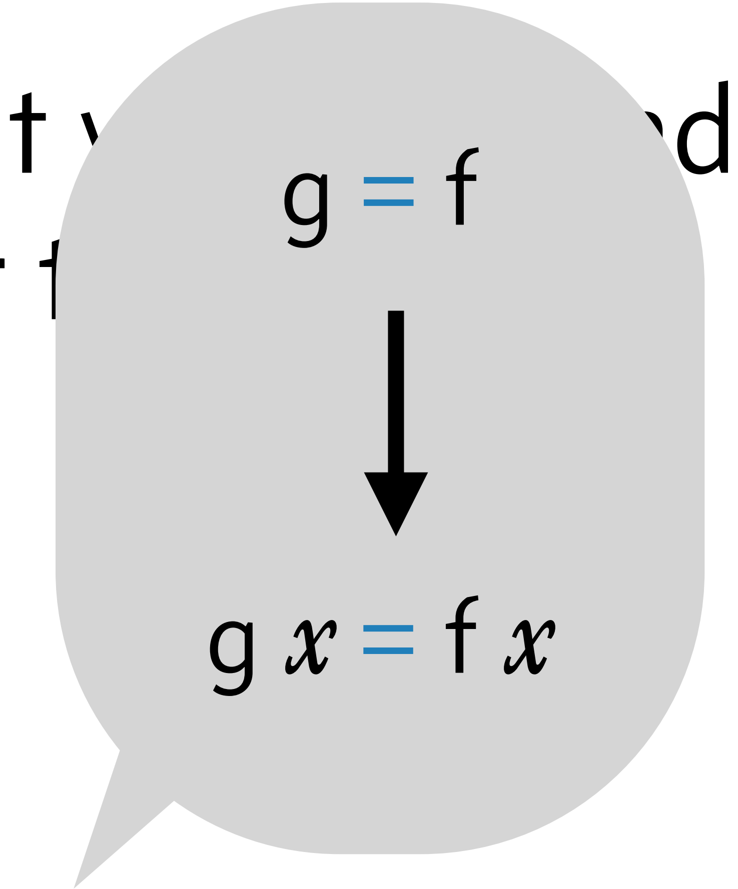

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Challenge 1bis: The Redundancy Criterion

Solution:
- Use weaker notion of entailment

Examples:
- b = a makes f b = f a redundant
- g = f **does not** make g a = f a redundant

Challenge 1bis: The Redundancy Criterion

Solution:

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- $g_0 = f_0$ **does not** make $g_1 a_0 = f_1 a_0$ redundant

Nonground Redundancy Criterion

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If a clause of the form

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Challenge 2: Unification

In first-order logic:

- Most general unifiers always exist and can be computed
  e.g. unifying f(a, y) with f(x, b) yields the mgu $\{x \mapsto \mathrm{a},\ y \mapsto \mathrm{b}\}$

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In higher-order logic:

A. Most general unifiers **do not** always exist
   e.g. unifying f (y a) and y (f a) yields infinitely many unifiers
   $\{y \mapsto \lambda x.\ x\}$ $\{y \mapsto \lambda x.\ \mathrm{f}\ x\}$ $\{y \mapsto \lambda x.\ \mathrm{f}\ (\mathrm{f}\ x)\}$ …

B. Unification is undecidable

C. Applied variables can hide positions where inferences should be made

Challenge 2: Unification

Solutions:

A. Use a (possibly infinite) sequence of unifiers instead of mgu
B. Interweave unification and inferences
C. Introduce a special "fluid" version of the main inference

Example: Fluid Inference

$\mathcal{F}$

$f\ a = c \qquad h\ (y\ b)\ (y\ a) \neq h\ (g\ (f\ b))\ (g\ c)$

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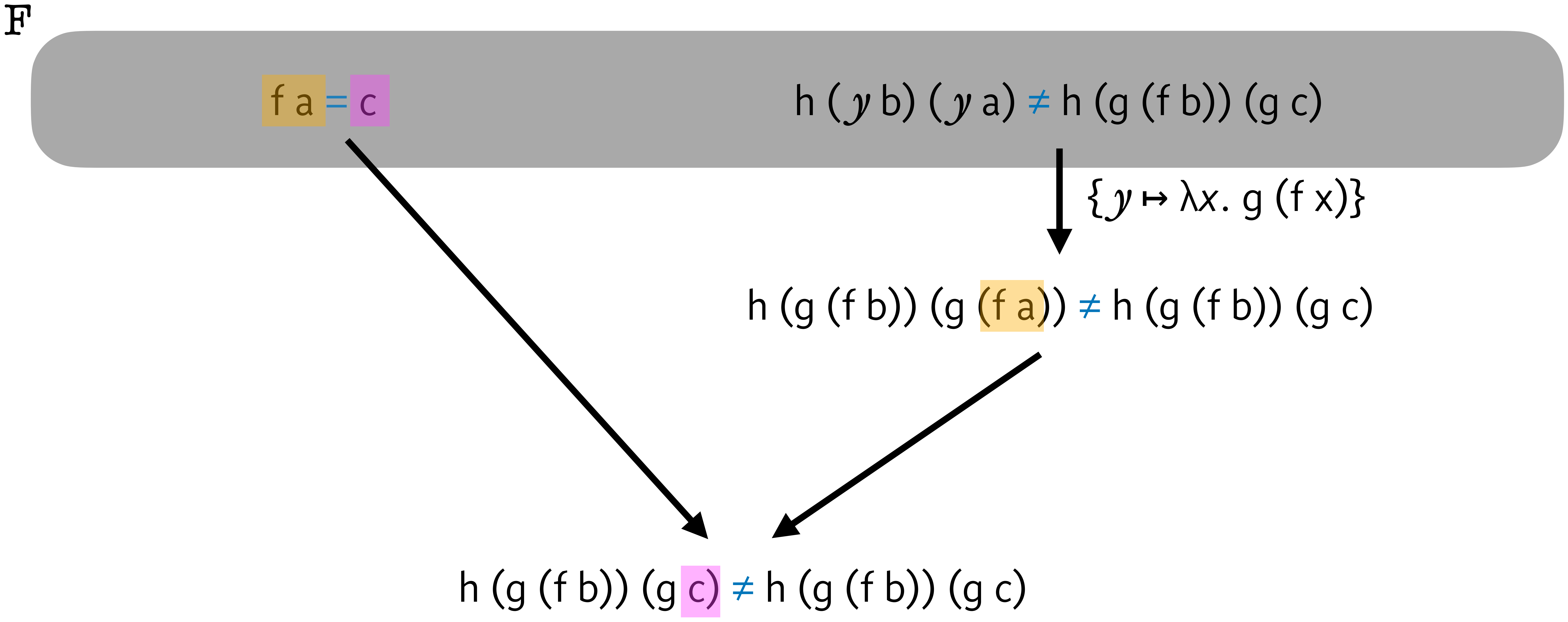

Example: Fluid Inference

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$z\,(\mathsf{f}\ \mathsf{a}) = z\ \mathsf{c}$

$\mathsf{h}\ (y\ \mathsf{b})\ (y\ \mathsf{a}) \neq \mathsf{h}\ (\mathsf{g}\ (\mathsf{f}\ \mathsf{b}))\ (\mathsf{g}\ \mathsf{c})$

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Challenge 3: Boolean Expressions

In first-order logic:

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- Clausification is simple and focuses on the outer skeleton

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Higher-order issues:

A. Formulas can appear nested in terms, including under λ's
   e.g. (λx. if $\exists y$. p x y then a else b)

B. We cannot perform clausification entirely in preprocessing

Challenge 3: Boolean Expressions

Solution:

- We introduce dedicated inference rules to clausify dynamically

Soundness of λ-Superposition

The inference rules are easy to show sound

In particular, the extra rule is justified by **argument congruence**:
if $[\![g]\!]^{\mathcal{J}} = [\![f]\!]^{\mathcal{J}}$, then $[\![g\ a]\!]^{\mathcal{J}} = [\![f\ a]\!]^{\mathcal{J}}$ for any a

Completeness of λ-Superposition

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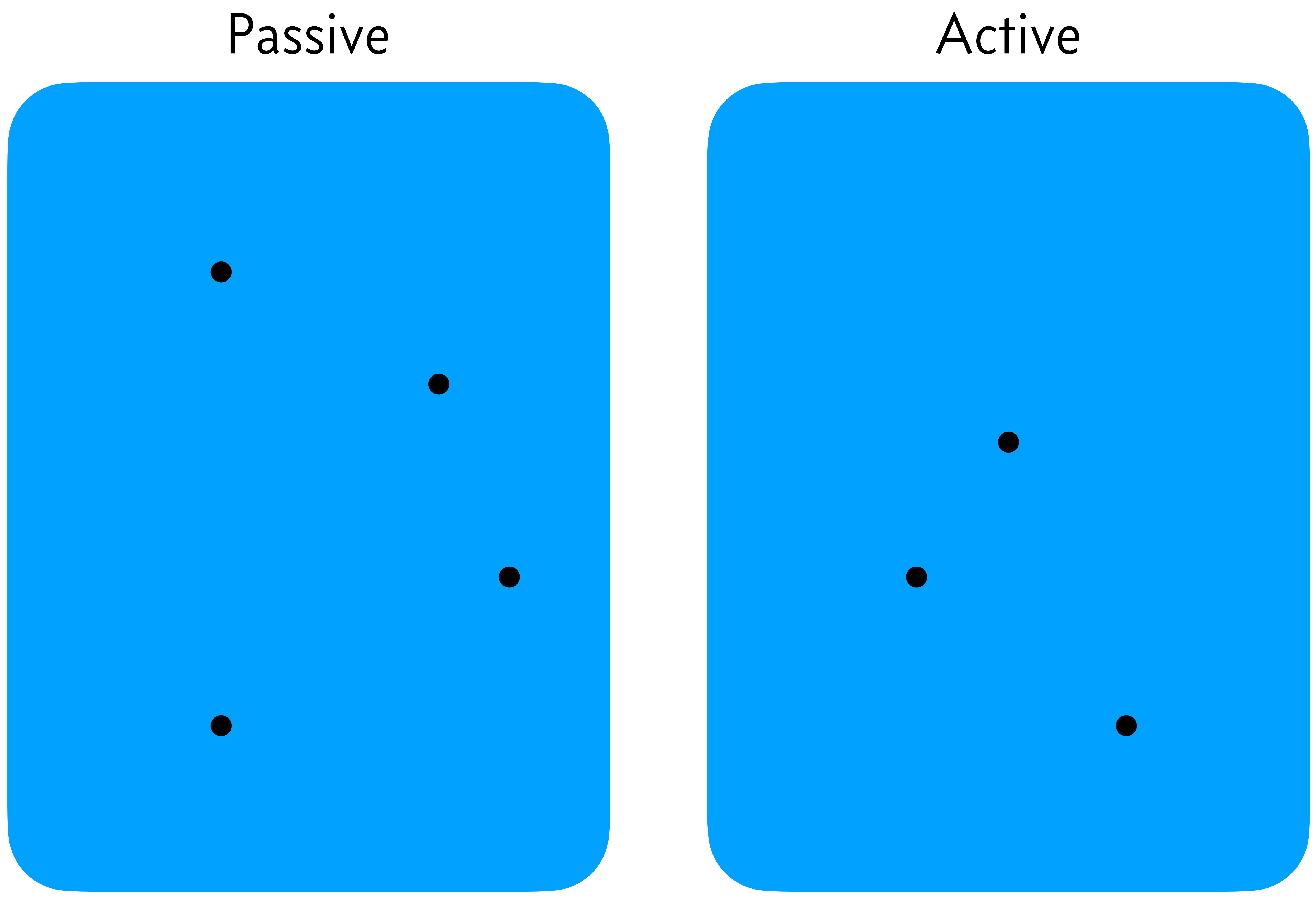
* with respect to the so-called Henkin semantics

Standard Saturation Loop



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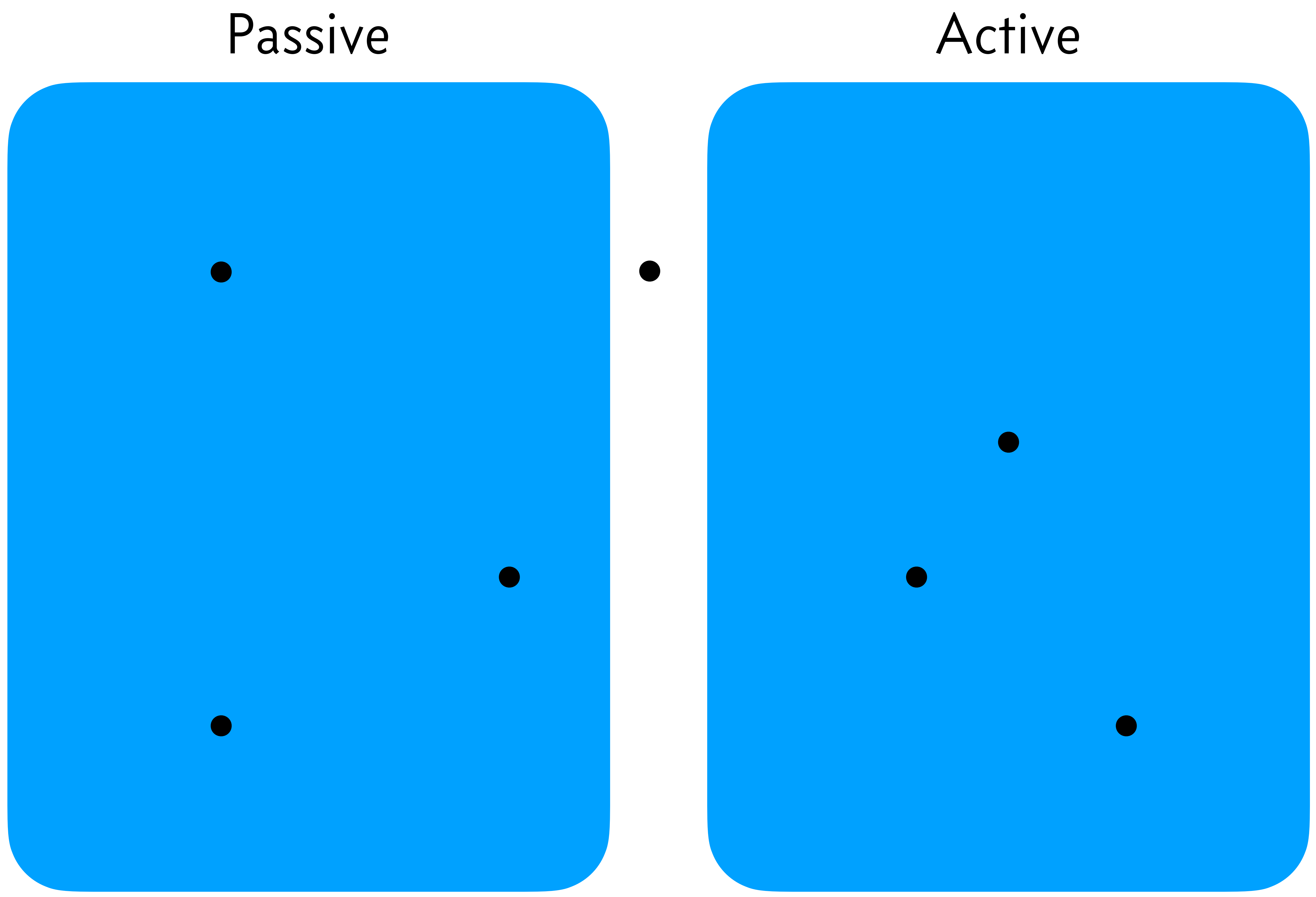

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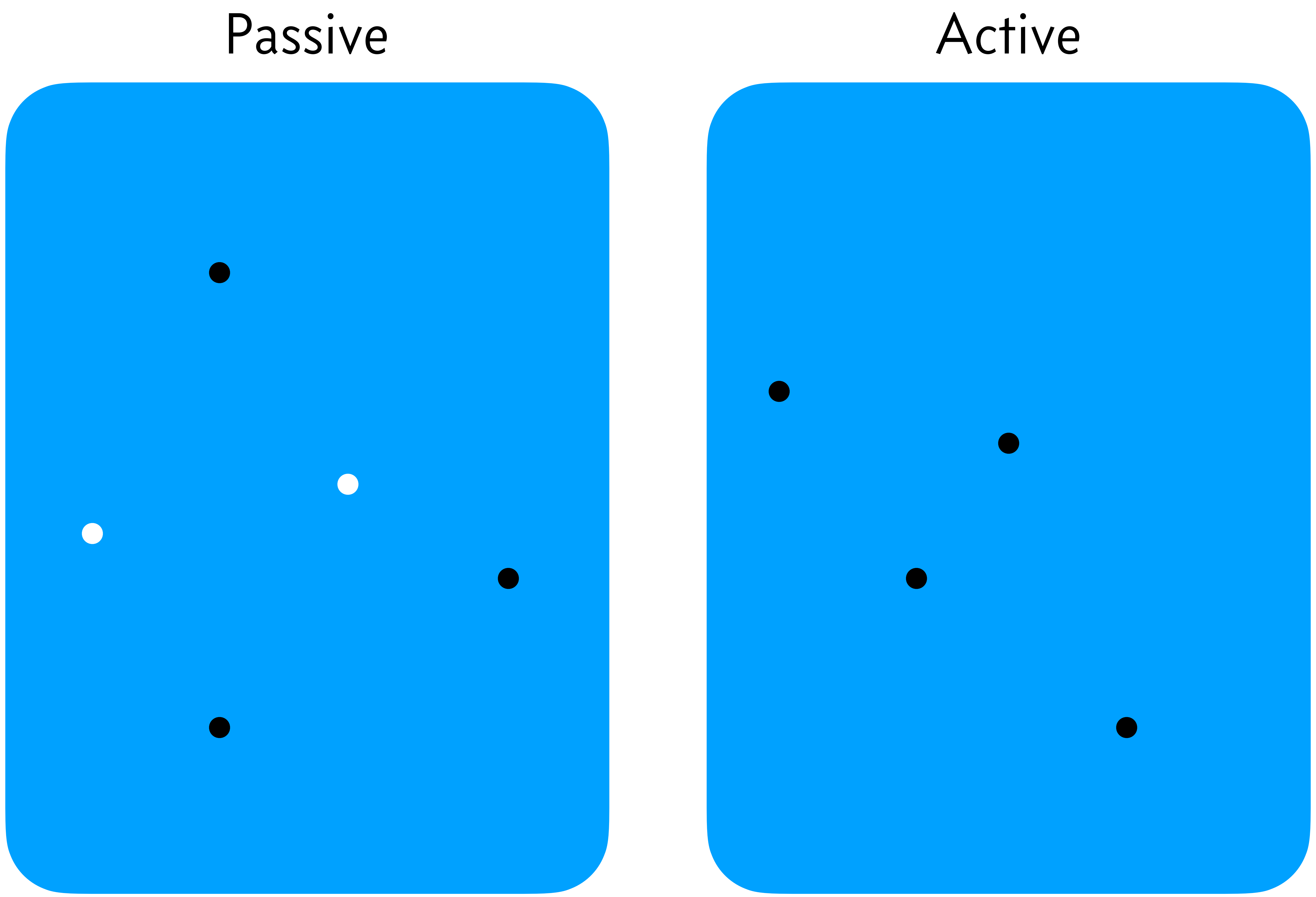
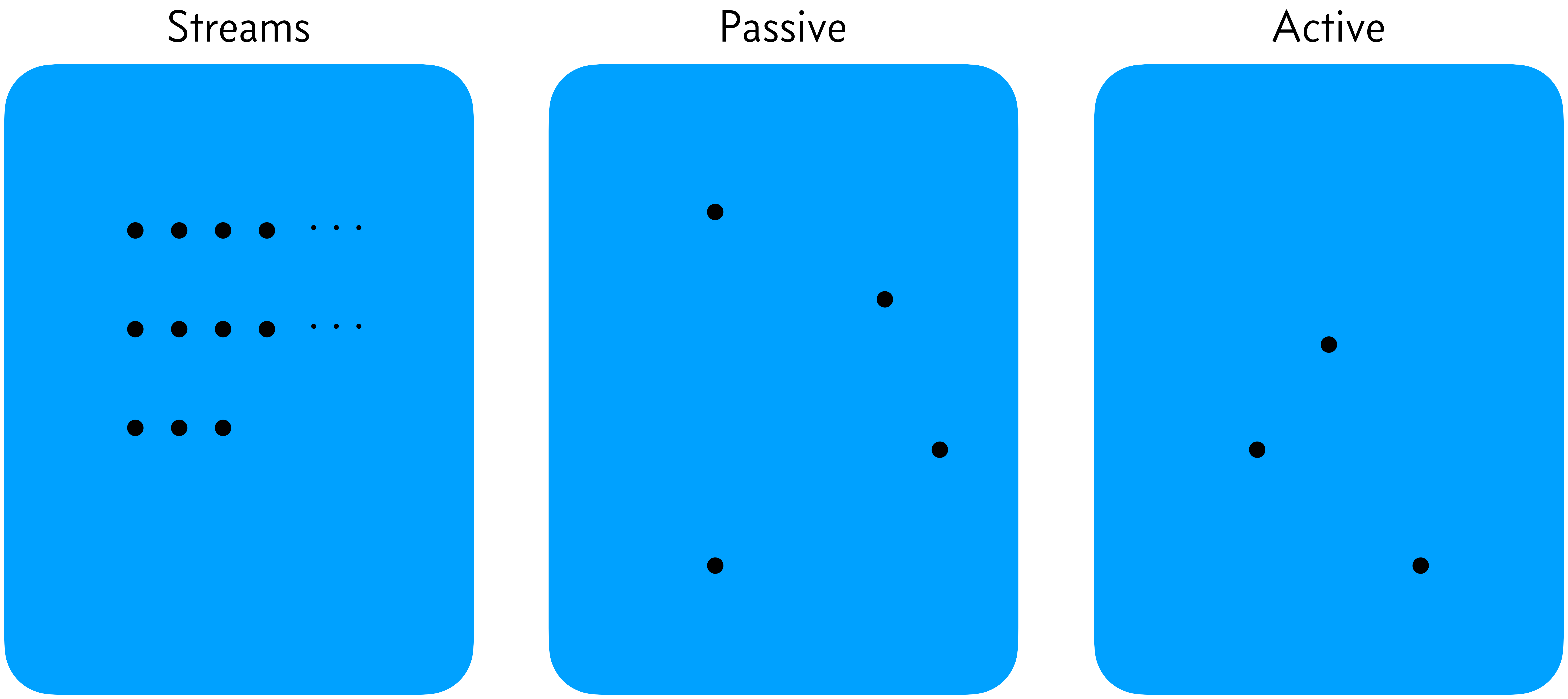


Generalized Saturation Loop



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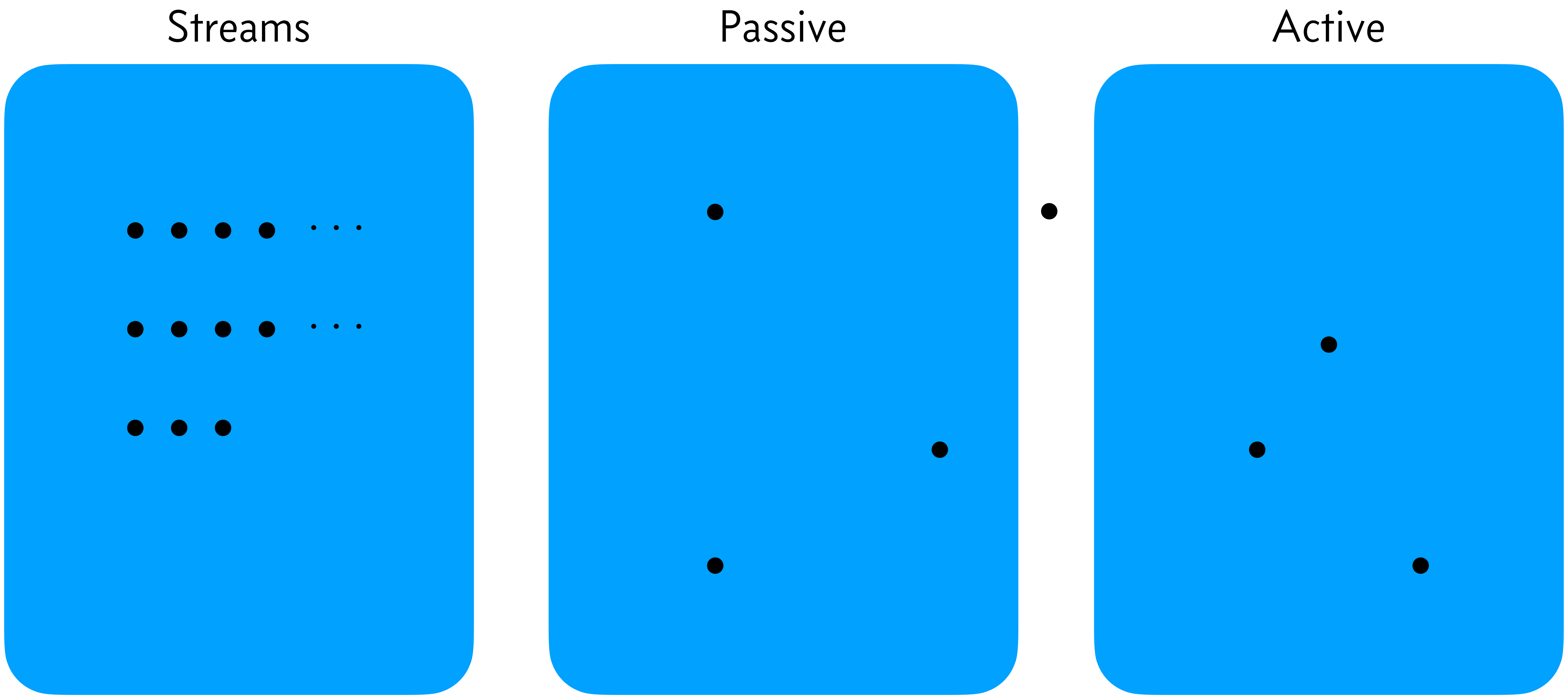

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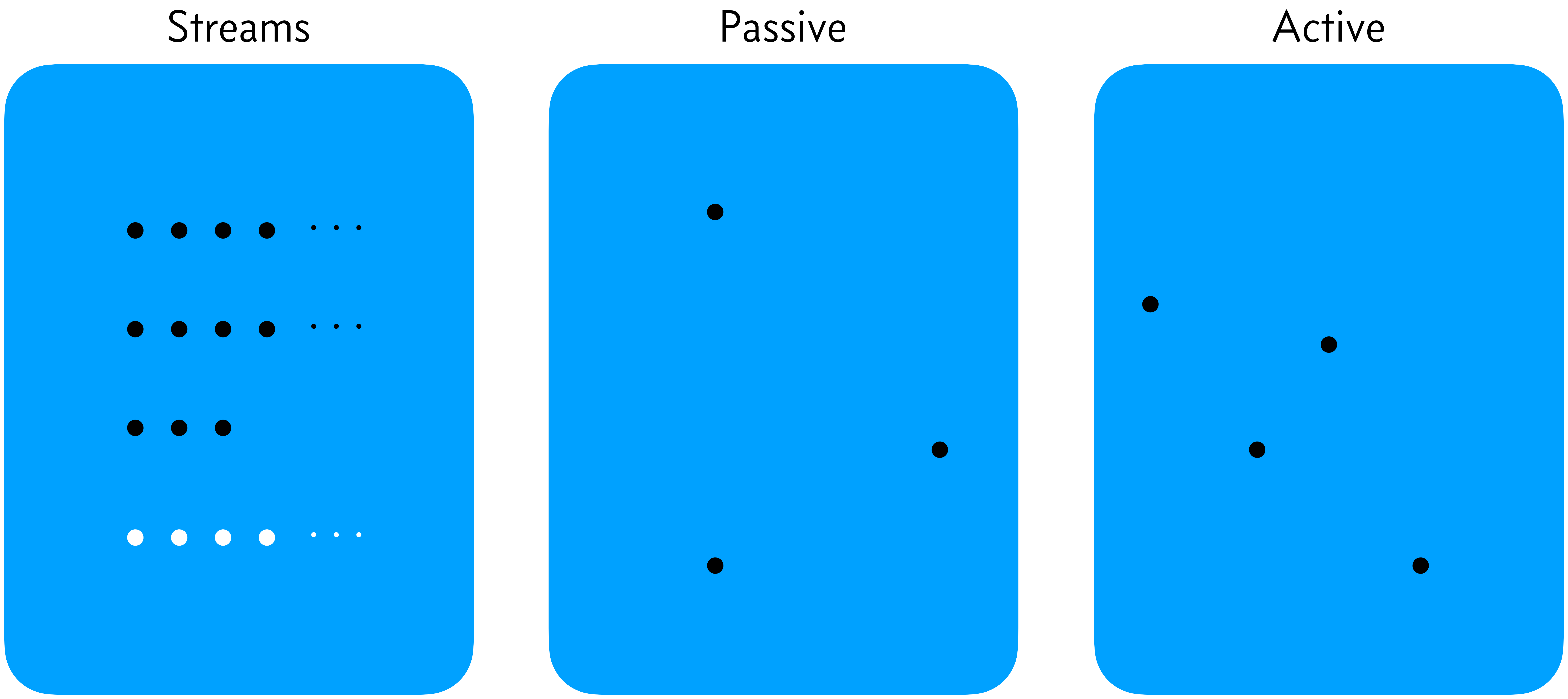

Competition Results (CASC 2023)

Higher-order Theorems	Vampire 4.8	Zipperpin 2.1.9999	*Zipperpin 2.1.999*	E 3.1	Leo-III 1.7.8	Satallax 3.4	cvc5 1.0.5	Lash 1.13	LEO-II 1.7.0	Duper 1.0
Solved/500	452/500	440/500	438/500	407/500	302/500	268/500	258/500	208/500	58/500	36/500
Solutions	452 90%	440 88%	438 87%	407 81%	302 60%	268 53%	258 51%	196 39%	58 11%	36 7%

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Solved/1000	467/1000	467/1000	462/1000	454/1000	362/1000	278/1000	219/1000	125/1000	51/1000
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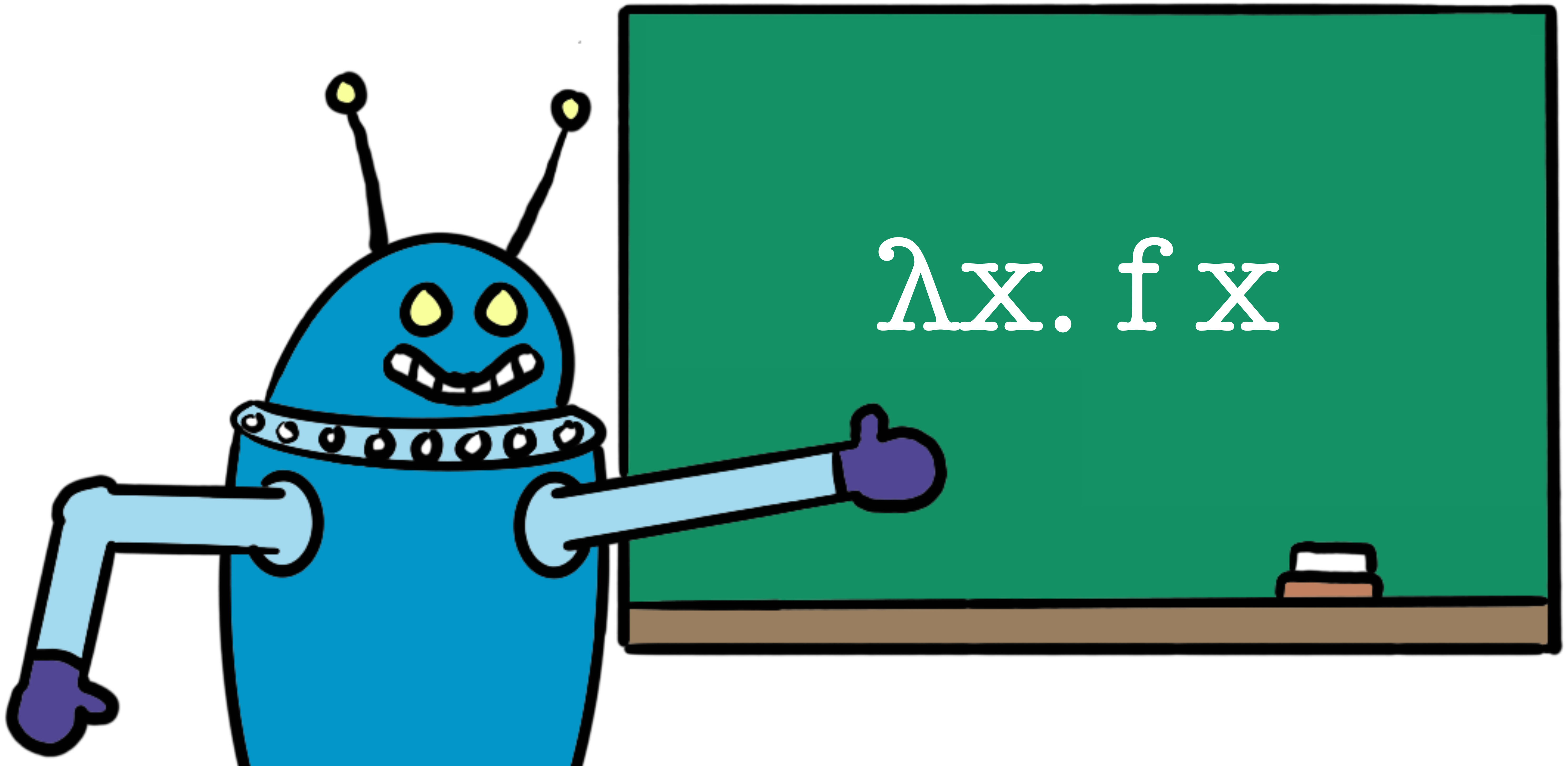
Provers & Solvers

Lecture 3: λ-Superposition



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Partly based on slides by
Alexander Bentkamp

Blah