# Verification of Linear Dynamical Systems

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# A Landscape

# Overview

Verification of linear systems (Markov chains, linear constraint loops, probabilistic and quantum automata, affine programs, linear recurrences, . . . )

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   - Termination for linear constraint loops

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1. **Halting Problem**:
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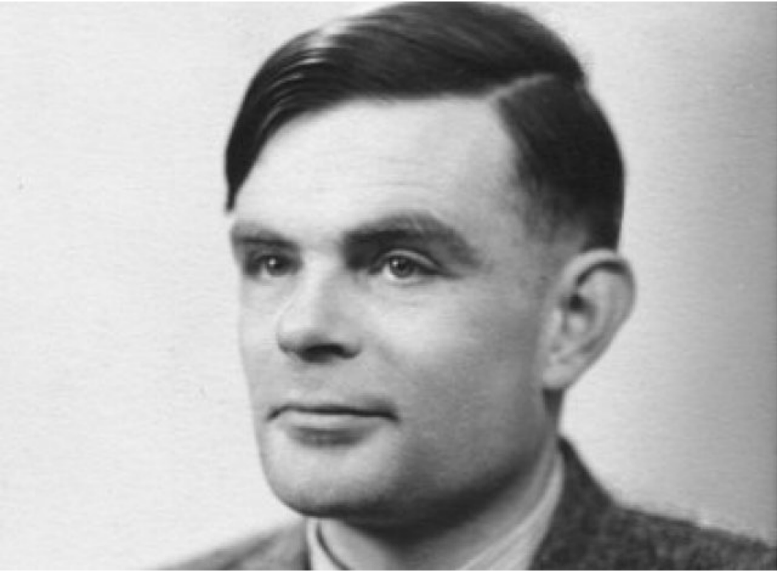
2. **Termination analysis**:
    - Termination for linear constraint loops
3. **Invariant synthesis**:
    - Computing polynomial invariants for affine programs

# Part I: Halting Problem



*What is the simplest class of programs for which decidability of the Halting Problem is open?*

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& y := 0; \\
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# Classical Formulation

A **linear recurrence sequence (LRS)** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ in $\mathbb{Q}$ such that there are constants $a_1, \ldots, a_k$ and, $\forall n \geq 0 : \quad u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n.$

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*Instance:* An LRS $\langle u_0, u_1, u_2, \ldots \rangle$

*Question:* Does $\exists n \geq 0$ such that $u_n = 0$?

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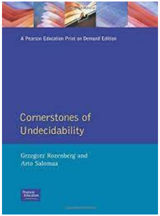

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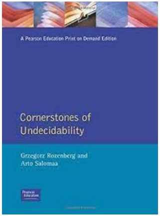


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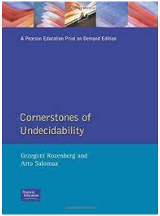


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**Problem ULTIMATE POSITIVITY**

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*Question*: Is $u_n \geq 0$ for all but finitely many $n$?

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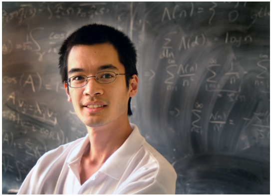

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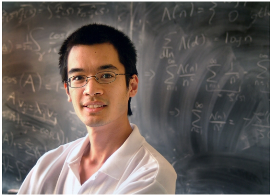

*"A mathematical embarrassment . . . "*

Richard Lipton

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- Decidability of the Skolem Problem is equivalent to being able to compute the finite set of zeros of any given non-degenerate LRS
- Unfortunately, all known proofs of the Skolem-Mahler-Lech Theorem make use of *non-constructive* $p$-adic techniques

# Quiz on Computational Complexity

- Given two NFA $A$ and $B$, is every word accepted by $A$ also accepted by $B$?

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- Given a Markov chain over states $s_1, \ldots, s_k$ with initial state $s_1$, is there some timepoint from which the probabiity to be in state $s_k$ is always greater than $1/2$?

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  - **ULTIMATE POSITIVITY-COMPLETE**

# Some Other Application Areas

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- Theoretical biology
  - analysis of L-systems
  - population dynamics
- Software verification / program analysis
- Dynamical systems
- Differential privacy
- (Weighted) automata and games
- Analysis of stochastic systems
- Control theory
- Quantum computing
- Statistical physics
- Formal power series
- Combinatorics
- ...

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Critical ingredient is Baker's theorem for linear forms in logarithms, which earned Baker the Fields Medal in 1970.

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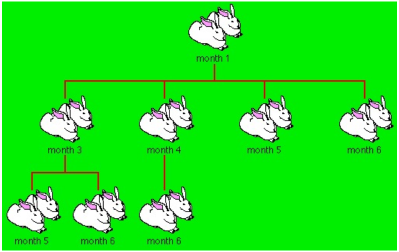
Simple LRS correspond precisely to **diagonalisable** matrices

# Counting Rabbits Modulo $m$

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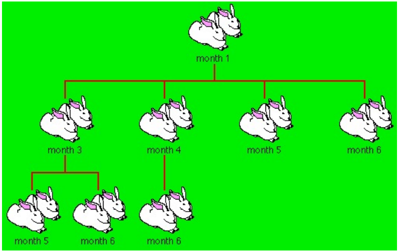


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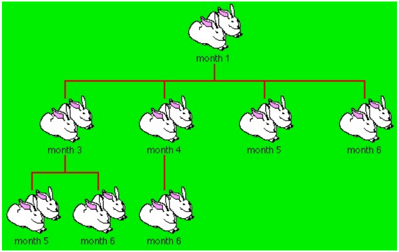


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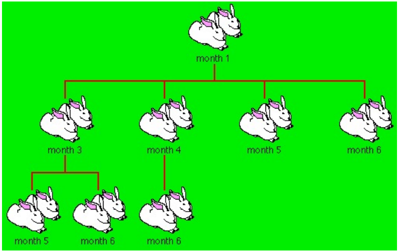


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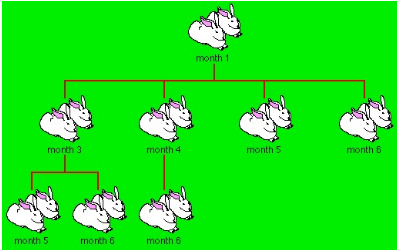


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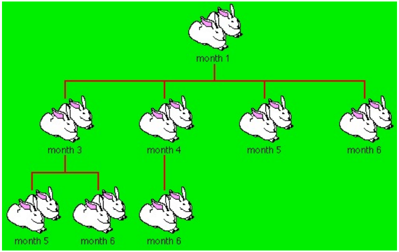


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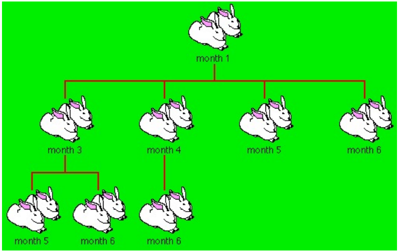
# Reversibility

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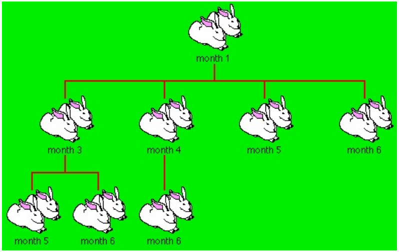


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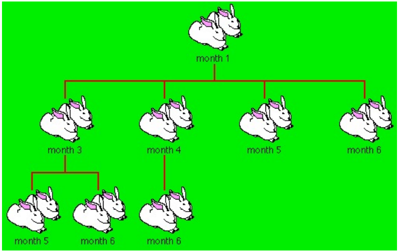


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  that contains a zero
- The bi-infinite extension is periodic modulo *m* for every *m*

# Skolem Conjecture

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VON

TH. SKOLEM

AVHANDLINGER UTGITT AV DET NORSKE VIDENSKAPS-AKADEMI I OSLO
I. MAT.-NATURV. KLASSE. 1937. No. 12

- If a **simple bi-infinite** LRS over the rationals has no zeros, then it has no zeros modulo *some* integer $m$.

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- Why simple? Consider $u_n = 2^n(2n+1)$

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TH. SKOLEM

AVHANDLINGER UTGITT AV DET NORSKE VIDENSKAPS-AKADEMI I OSLO
I. MAT.-NATURV. KLASSE. 1937. No. 12

- If a **simple bi-infinite** LRS over the rationals has no zeros, then it has no zeros modulo *some* integer $m$.
- Why simple? Consider $u_n = 2^n(2n+1)$
- One-way version fails to hold

# Skolem Problem for Bi-Infinite Sequences

**Problem BI-SKOLEM**

*Instance*: A bi-LRS $\langle \ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots \rangle$ over $\mathbb{Q}$

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- Decidable for simple LRS assuming Skolem's Conjecture.
- How are the Skolem and Bi-Skolem Problems related?
- Can one use an oracle for Bi-Skolem to compute all zeros of a bi-LRS?

# Reducing Skolem to Bi-Skolem

**Theorem (Bilu, Luca, Pursar, Ouaknine, Nieuwveld, W. 22)**

*For LRS of order 5 the Skolem and Bi-Skolem Problems are interreducible.*

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**Lemma (Zero Isolation)**

*Assuming the p-adic Schanuel Conjecture, given a bi-infinite LRS $\langle u_n \rangle_{n=-\infty}^{\infty}$ one can compute $L$ such that $u_{Ln} \neq 0$ for all $n \neq 0$.*

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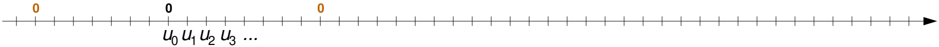

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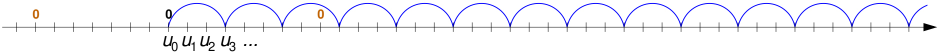

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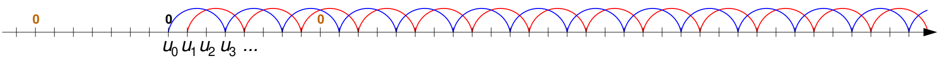

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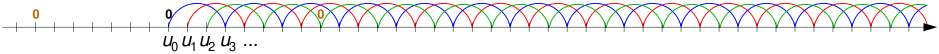

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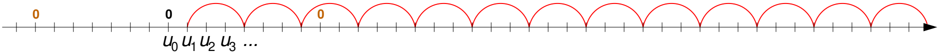

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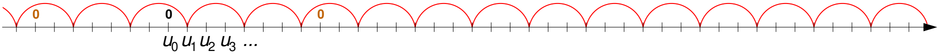

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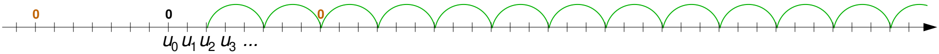

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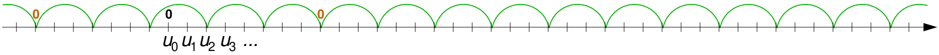

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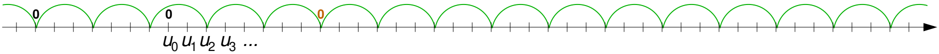

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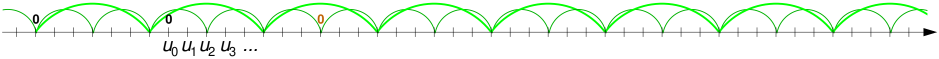

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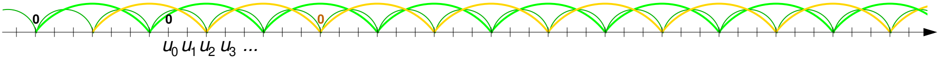

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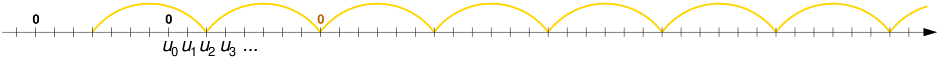

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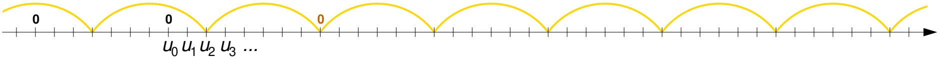

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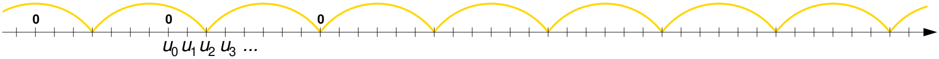

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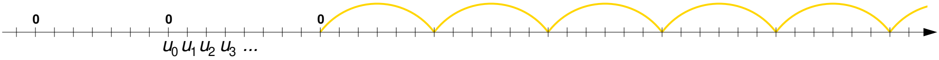

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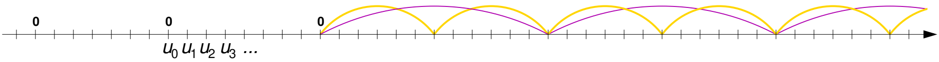

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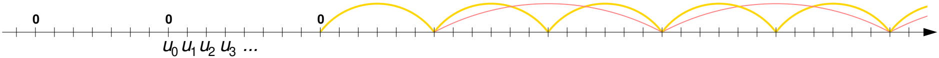

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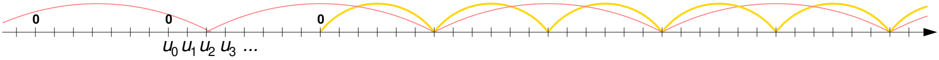

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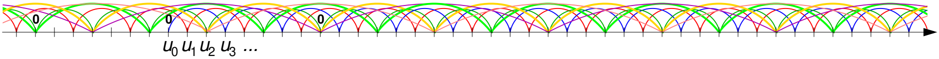

# Schanuel's Conjecture

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Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then $\{\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\}$, contains (at least) $n$ numbers that are algebraically independent over $\mathbb{Q}$.

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In other words: for any polynomial $P(x_1, \ldots, x_n)$ with rational (or algebraic) coefficients, if $P(\beta_1, \ldots, \beta_n) = 0$, then $P$ must be the zero polynomial.

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If *both* $e + \pi$ and $e\pi$ were rational, then $e$ and $\pi$ would be algebraic, contradiction.

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Thus for *any* non-zero polynomial $P(x, y)$ with rational (or algebraic) coefficients, we have that $P(e, \pi)$ cannot be zero.

Therefore $e + \pi$, $e\pi$, and $e^5\pi^3 - e^2\pi^7 + e$ must all be irrational (in fact, transcendental).

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**Lemma (Zero Isolation)**

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- There is a punctuated disk around zero in which $f$ is non-zero.

# Decision Procedure for the Skolem Problem

**Theorem (Bilu, Luca, Nieuwveld, Ouaknine, Pursar, W., 22)**

*There is a decision procedure for the Skolem Problem for simple LRS that terminates subject to the p-adic Schanuel Conjecture and the Skolem Conjecture.*

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- If a zero is found, use Zero-Isolation Lemma to split input into subsequences and then recurse
- Output is a list of zeroes and certificate that there are no more zeroes

skolem.mpi-sws.org

Accounts Teams

# SKOLEM: Solves the Skolem Problem for simple integer LRS

## System Explanation Show/Hide

- On the first line write the coefficients of the recurrence relation, separated by spaces.
- On the second line write an equal number of space-separated initial values.
- The LRS must be simple, non-degenerate, and not the zero LRS.
- The tool will output all zeros (at both positive and negative indices), along with a completeness certificate.

## Input Format

$a_1\ a_2\ \dots\ a_k$

$u_0\ u_1\ \dots\ u_{k-1}$

where:

$u_{n+k} = a_1 \cdot u_{n+k-1} + a_2 \cdot u_{n+k-2} + \dots + a_k \cdot u_n$

## Input area

Auto-fill examples: Show/Hide

Zero LRS | Degenerate LRS | Non-simple LRS | Trivial | Fibonacci | Tribonacci | Berstel sequence [1] | Order 5 [3] | Order 6 [3] | Reversible order 8 [3]

Manual input:

```
6 -25 66 -120 150 -89 18 -1
0 0 -48 -120 0 520 624 -2016
```

- Always render full LRS (otherwise restricted to 400 characters)
- I solemnly swear the LRS is non-degenerate (skips degeneracy check, it will timeout or break if the LRS is degenerate!)
- Factor subcases (merges subcases into single linear set, sometimes requires higher modulo classes)
- Use GCD reduction (reduces initial values by GCD)
- Use fast identification of mod-m (requires GCD reduction) (may result in non-minimal mod-m argument)

Go Clear Stop

## Output area

Zeros: 0, 1, 4

Zero at 0 in $(0+1\mathbb{Z})$ hide/show

- p-adic non-zero in $(0+136\mathbb{Z}_{\neq 0})$
- Zero at 1 in $(1+136\mathbb{Z})$ hide/show
  - p-adic non-zero in $(1+680\mathbb{Z}_{\neq 0})$ $((0+5\mathbb{Z}_{\neq 0})$ of parent)
  - Non-zero mod 3 in $(137+680\mathbb{Z})$ $((1+5\mathbb{Z})$ of parent)
  - Non-zero mod 3 in $(273+680\mathbb{Z})$ $((2+5\mathbb{Z})$ of parent)
  - Non-zero mod 9 in $(409+680\mathbb{Z})$ $((3+5\mathbb{Z})$ of parent)
  - Non-zero mod 3 in $(545+680\mathbb{Z})$ $((4+5\mathbb{Z})$ of parent)
- Non-zero mod 7 in $(2+136\mathbb{Z})$

```
================
LRS: u_{n} =
-271613116171209744858663520558946347040150955089064191363633545467540976915
1} +
-508757179425530608464927613320696582397187501636529439512475357072393244955
2} +
-102066400158641189915199426519447202492215998409667435547930568677820080526
3} +
-141209566240600031036449671518126066729890157506482293126851759080465437598
4} +
190695589477320710360984265894091422375694233909158701965446106943727346702
5} +
```