Part II: Loop Termination

In contrast to popular belief, proving termination is not always impossible.

BY BYRON COOK, ANDREAS PODELSKI, AND ANDREY RYBALCHENKO

Proving Program Termination

[...] termination tools can automatically prove or disprove termination of many famous complex examples such as Ackermann's function or McCarthy's 91 function as well as moderately sized industrial examples

. Complete Method for the Synthesis of Linear Ranking Functions

Andreas Podelski and Andrey Rybalchenko

Max-Planck-Institut für Informatik
Saarbrücken, Germany

"We present an automated method for proving the termination of an unnested loop by **synthesizing linear ranking functions***. The method is used as a subroutine for proving termination of more general programs [. . .] "*

Linear Constraint Loops - A Useful Abstraction

Complete method for synthesising linear and lexicographic-linear ranking functions for linear constraint loops.

$$\textit{while } (B\boldsymbol{x} \geq \boldsymbol{b}) \textit{ do } A\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}'\end{pmatrix} \leq \boldsymbol{c}$$

Termination Detection @ PODC'91

Termination Detection in Logic Programs using Argument Sizes*

(Extended Abstract)

Kirack Sohn and Allen Van Gelder
University of California, Santa Cruz

"The prospects for automatic termination detection of logic programs appear promising, in constrast to the **bleak picture for procedural languages***. We describe a method of finding a non-negative linear combination of bound arguments that decreases during top-down execution of the recursive rules."*

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A new look at the automatic synthesis of linear ranking functions ☆

Roberto Bagnara [a,b,∗], Fred Mesnard [c], Andrea Pescetti [d], Enea Zaffanella [a,b]

[a] Dipartimento di Matematica, Università di Parma, Italy

"In this paper we present two algorithms, one based on work by Sohn and Van Gelder; the other, due to Podelski and Rybalchenko. [We show that] **the two algorithms synthesize a linear ranking function under exactly the same set of conditions**.*"*

Linear Constraint Loops - Decidability

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"We feel that the **most intriguing problem** *is whether the termination of a single linear constraint loop is decidable, when the variables range over the integers."*

Ben-Amram, Genaim, and Masud,
On the Termination of Integer Linear Loops
ACM Trans. Program. Lang. Syst., 2012

Polya's Advice

"If there's a hard problem you can't solve, there's an easier problem you can solve: find it!"

Deterministic Linear Loops

while $\mathbf{A}\mathbf{x} \geq \boldsymbol{b}$ do
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Termination Problem (R)

Instance: $\langle\ \mathbf{A},\ \mathbf{B},\ \mathbf{b},\ \mathbf{c}\ \rangle$
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Termination over $\mathbb{R}$ is decidable.

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Termination over $\mathbb{R}$ is decidable.

Theorem (Braverman 08)

Termination over $\mathbb{Q}$ is decidable.

Termination Depends on the Numerical Domain

Loop that is terminating over $\mathbb{Q}$ but not $\mathbb{R}$:

$$\begin{array}{rl} \textbf{while} & 4x + y > 0 \\ \textbf{do} & \begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} -2 & 4 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

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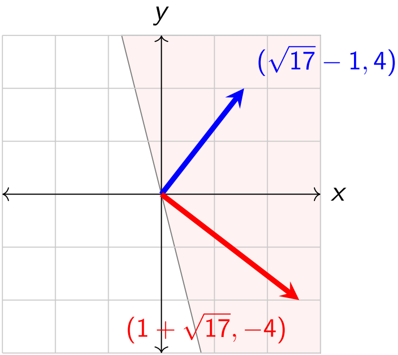

Termination of Deterministic Linear Loops over $\mathbb{Z}$

" It appears that to decide termination over $\mathbb{Z}$ it is necessary to be able to tell, given a point $\boldsymbol{x}_0$, whether the program terminates on $\boldsymbol{x}_0$ or not."

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Halting Problem

Instance: $\langle\ \mathbf{A}, \mathbf{B}, \mathbf{b}, \mathbf{c}, \boldsymbol{x}_0\ \rangle$
Question: Does the loop halt when started on $\boldsymbol{x}_0$?

Linear Recurrence Sequences

The sequence of values assumed by a variable in a deterministic linear loop is a **linear recurrence sequence**: $\langle u_n \rangle_{n \geq 0}$:

$$u_n = a_1 u_{n-1} + \cdots + a_k u_{n-k} \quad (n \geq k)$$

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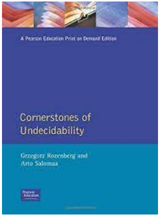
\end{aligned}$$

- Conversely, if u_n satisfies the above recurrence then

$$u_n = \begin{pmatrix} 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_{k-1} & a_{k-2} & \cdots & a_0 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \end{pmatrix}^n \begin{pmatrix} u_{k-1} \\ \vdots \\ u_1 \\ u_0 \end{pmatrix}$$

The Positivity Problem



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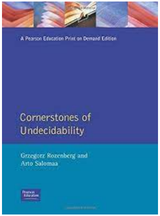

Problem POSITIVITY

Instance: An LRS $\langle u_0, u_1, u_2, \ldots \rangle$
Question: Is $u_n \geq 0$ for all n?

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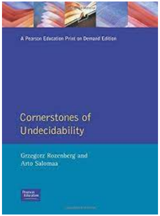

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Problem ULTIMATE POSITIVITY

Instance: An LRS $\langle u_0, u_1, u_2, \ldots \rangle$

Question: Is $u_n \geq 0$ for all but finitely many n?

The Many Faces of the Positivity Problem

Positivity of Rational Series

Instance: Rational Series $f(x) = \dfrac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} a_n x^n$

Question: Are all coffecients of f postive ?

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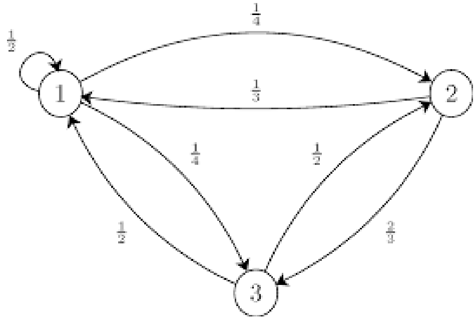
Threshold Problem for Markov Chains

Instance: Markov chain M, distinguished state s, and threshold λ.

Question: Is the probability of being in state s at least λ at all times ?

The Many Faces of the Positivity Problem



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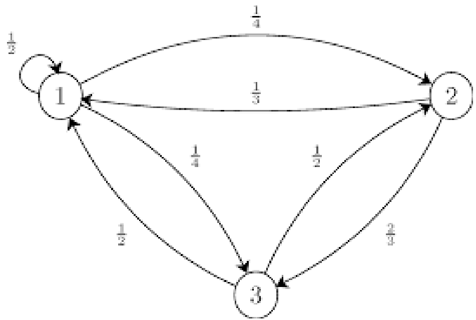

- Is the probability of being in initial state > 0.455 at all times ?

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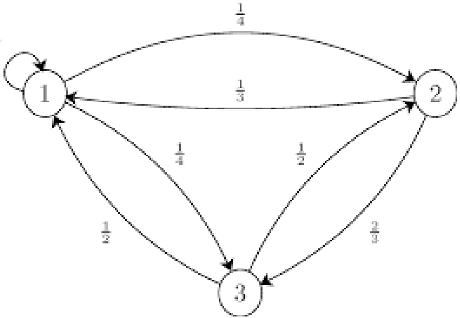

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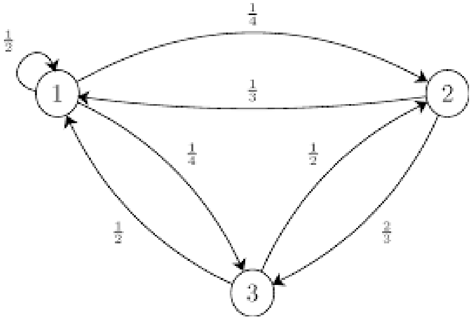

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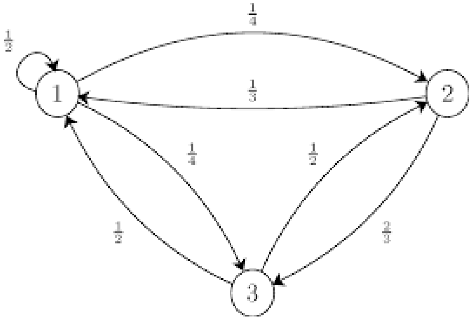

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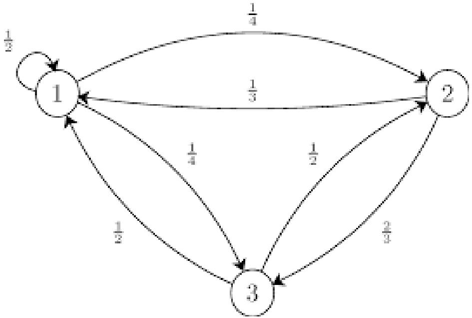

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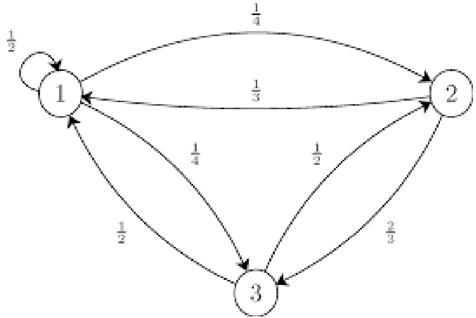

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(\mathbf{0.457}, 0.255, 0.288)
\end{array}
$$

Closed Forms for Numerical Loops*

ZACHARY KINCAID, Princeton University, USA
JASON BRECK, University of Wisconsin, USA
JOHN CYPHERT, University of Wisconsin, USA

"It is well-known that a **closed-form representation** *of the behavior of a linear dynamical system can always be expressed using algebraic numbers, but this approach can create formulas that present an* **obstacle for automated-reasoning tools**.*"*

Exponential Polynomial Solutions

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Theorem

There exist polynomials C_1, ..., C_m such that, for all n,

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Question

With the closed form in hand, can automated tools and decision procedures help decide positivity?

Deciding Positivity - Case Study

Determine positivity of LRS

$$u_n := \frac{33}{8} + \lambda_1^n + \overline{\lambda_1^n} + 2\lambda_2^n + \overline{2\lambda_2^n},$$

where $\lambda_1 = \frac{-3+4i}{5}$ and $\lambda_2 = \frac{-7+24i}{25}$

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Lemma

If $\langle u_n \rangle_{n=0}^{\infty}$ is a simple LRS all of whose characteristic roots have the same absolute value then we can decide positivity.

Taking the Closure

Define $f : \mathbb{T}^2 \to \mathbb{R}$ by

$$f(z_1, z_2) = \tfrac{33}{8} + z_1 + \overline{z_1} + 2z_2 + 2\overline{z_2}\,.$$

Then $u_n = f(\lambda_1^n, \lambda_2^n)$.

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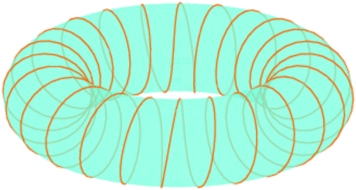
$$\mathrm{Cl}\{(\lambda_1^n, \lambda_2^n) : n \in \mathbb{N}\} = \underbrace{\{(z_1, z_2) \in \mathbb{T}^2 : z_1^2 z_2 = 1\}}_{S}$$



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Is $v_n := u_n - \frac{1}{2^{n+100}}$ positive ?

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Effective bounds from Baker's Theorem (1966) on linear forms in logarithms

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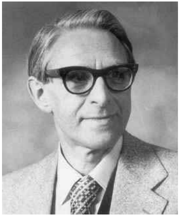
Non-effective bounds from the Subspace Theorem — a higher-dimensional generalisation of Roth's Theorem (1955) on Diophantine Approximation.



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Theorem (Ouaknine, W. 2014)

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Corollary

The Halting Problem is decidable for deterministic linear loops with at most 4 variables and for loops with at most 8 variables if the update matrix is diagonalisable.

Back to Termination Over $\mathbb{Z}$

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Define $\mathrm{NT} := \{\boldsymbol{x}_0 \in \mathbb{R}^n : \boldsymbol{x}_0 \text{ non-terminating}\}$

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Proposition

There exists an closed convex semi-algebraic loop invariant C s.t.

- $\mathrm{int}(C) \subseteq \mathrm{NT} \subseteq C$
- NT *contains an integer point iff C contains an integer point.*

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Question

Is it decidable whether a semi-algebraic set contains an integer point?

Flatness Theorem

Given convex $C \subseteq \mathbb{R}^d$, define

$$\mathrm{width}(C) := \inf_{\boldsymbol{v} \in \mathbb{Z}^d \setminus \{0\}} \sup_{\boldsymbol{x}, \boldsymbol{y} \in C} \boldsymbol{v}^\top (\boldsymbol{x} - \boldsymbol{y}) \,.$$

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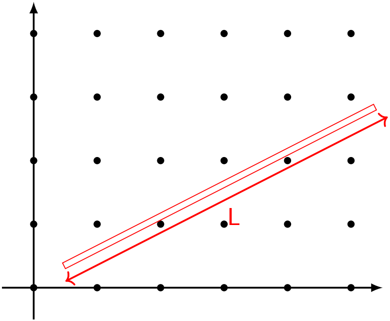

How does the lattice width vary as a function of L?

Hilbert's Tenth Problem for Convex Sets

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- **Halting** (i.e., the Positivity Problem for LRS) remains beyond reach, even in deterministic case.